

$$\mathbf{Fun}^{\text{ps}}(\mathcal{G}, \mathbf{Gpd}) = \mathbf{Gpd}/\mathcal{G}$$

$$\mathbf{Fun}(X, \mathcal{S}) = \mathcal{S}/X$$

$$\mathbf{Fun}^i(\mathcal{C}, \mathbf{AdjCat}) \stackrel{?}{=} \mathbf{AdjCat}/\mathcal{C}$$

(∞, ∞) -categories with adjoints are a bit like spaces

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III Encontro Brasileiro de Teoria das Categorias (2026)

A dictionary

Classical Mathematics	Homotopy Theory	Higher Category Theory
equality	homotopy	morphism
sets $\text{Set} = \text{Cat}_{(0,0)}$	spaces/ ∞ -groupoids $\mathcal{S} = \text{Cat}_\infty$	(∞, ∞) -categories Cat_∞
—	homotopy n -type	(∞, n) -category
*	disks \mathbb{D}^n	globes $\vec{\mathbb{D}}^n$
\emptyset	spheres S^n	$\partial\vec{\mathbb{D}}^{n+1}$ or $\vec{\mathbb{D}}^n/\partial\vec{\mathbb{D}}^n$
Cartesian product \times	Cartesian product \times	Gray tensor product \otimes
abelian groups Ab	spectra Sp	categorical spectra CatSp
—	$\mathbb{E}_\infty\mathcal{S}_{\text{gp-like}} \simeq \text{Sp}_{\geq 0}$	$\mathbb{E}_\infty\text{Cat}_\infty \simeq \text{CatSp}_{\geq 0}$
—	loop $\Omega(X, x) = \text{Aut}_X(x)$	$\Omega(X, x) = \text{End}_X(x)$
—	suspension $\Sigma = \mathcal{S}^1 \wedge (-)$	$\Sigma = \text{BFree}_{\mathbb{E}_1} = \text{BN} \circledast (-)$
—	homology $\simeq \text{Sp}$	categorical homology $\simeq \text{CatSp}$

Building blocks for ∞ -categories

HAVE:

- The large category of small categories CAT .
- The cartesian closed 1-category $\mathcal{S} =: \text{Cat}_0 \in \text{CAT}$.
- Enrichment over $(\mathcal{V}, \otimes) \rightsquigarrow \mathcal{V}\text{Cat} \in \text{CAT}$.
- Monoidal left (or right) adjoint $\mathcal{V} \rightleftarrows \mathcal{W} \rightsquigarrow \mathcal{V}\text{Cat} \rightleftarrows \mathcal{W}\text{Cat}$.

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GET:

- $\text{Cat}_{(n+1)} := \text{Cat}_n \text{Cat}$.
- $\text{Cat}_n \hookrightarrow \text{Cat}_{n+1}$ together with both adjoints.

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SET:

- $\text{Cat}_\infty := \lim_{\text{CAT}} (\text{Cat}_{n+1} \rightarrow \text{Cat}_n)$

Gray tensor product

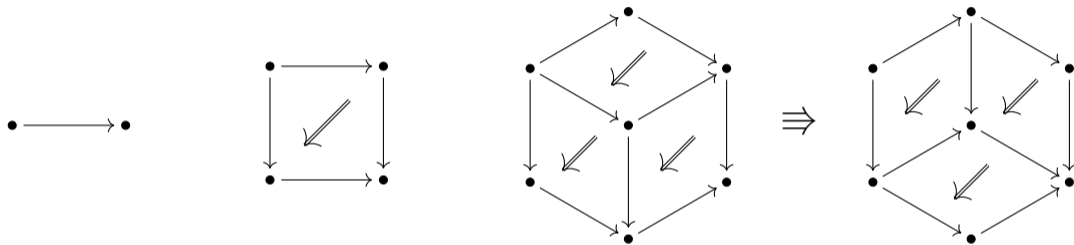
If $X \in \mathcal{S}$:

$$\begin{array}{ccc} X(x, y) & \longrightarrow & \text{Fun}(\overrightarrow{\mathbb{D}}^1, X) \\ \downarrow & \lrcorner & \downarrow \\ * & \xrightarrow{(x,y)} & X \times X \end{array}$$

But if $\mathcal{C} \in \text{Cat}_\infty$:

$$\begin{array}{ccc} \mathcal{C}(x, y)^\times & \longrightarrow & \text{Fun}(\overrightarrow{\mathbb{D}}^1, \mathcal{C}) \\ \downarrow & \lrcorner & \downarrow \\ * & \xrightarrow{(x,y)} & \mathcal{C} \times \mathcal{C} \end{array}$$

Gray tensor product



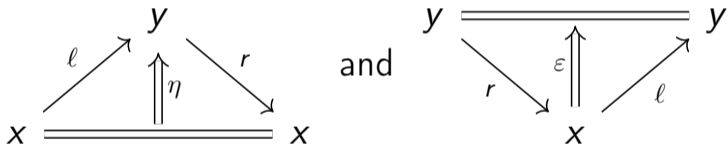
Theorem (Campion)

These cubes are dense in Cat_∞ , and extend to a unique closed monoidal structure

$$- \otimes \mathcal{C} : \text{Cat}_\infty \rightleftarrows \text{Cat}_\infty : \text{Fun}^{\text{lax}}(\mathcal{C}, -)$$

Adjoint in a (2,2)-category

An **adjunction** for a pair $\ell : x \rightleftarrows y : r$ in a (2,2)-category is a pair of 2-cells



satisfying $(\varepsilon.f)(f.\eta) = 1_f$ and $(g.\varepsilon)(\eta.g) = 1_g$.

Adjoints in an ∞ -category

walking adjunction $\rightsquigarrow \mathbf{adj} \in \mathbf{Cat}_{(2,2)}$

walking “homotopy coherent adjunction” $\rightsquigarrow \mathbf{adj}^{\simeq} \in \mathbf{Cat}_2 \hookrightarrow \mathbf{Cat}_\infty$

Theorem (Riehl–Verity, Masuda’s interpretation)

The maps $\overrightarrow{\mathbb{D}}^1 \xrightarrow{\ell \text{ or } r} \mathbf{adj}$ are epimorphisms, and $\mathbf{adj} \hookrightarrow \mathbf{adj}^{\simeq}$ is an equivalence.

Categories with adjoints

$\vec{\mathbb{D}}^1 \xrightarrow{\ell \text{ or } r} \mathbf{adj}$ are epimorphisms

\rightsquigarrow the ℓ -local objects

$$\begin{array}{ccc} \vec{\mathbb{D}}^1 & \longrightarrow & \mathcal{C} \\ \ell \downarrow & \nearrow & \\ \mathbf{adj} & & \end{array} !$$

are the ∞ -categories with left adjoints for 1-morphisms.

$$\mathbf{Adj}_{\leq 1} \mathbf{Cat}_{\infty} = \mathbf{Cat}_{\infty}[\{\ell, r\}^{-1}] \rightleftarrows \mathbf{Cat}_{\infty}$$

Categories with all adjoints

Recall the suspension $\sigma : \text{Cat}_\infty \rightarrow \text{Cat}_\infty$:

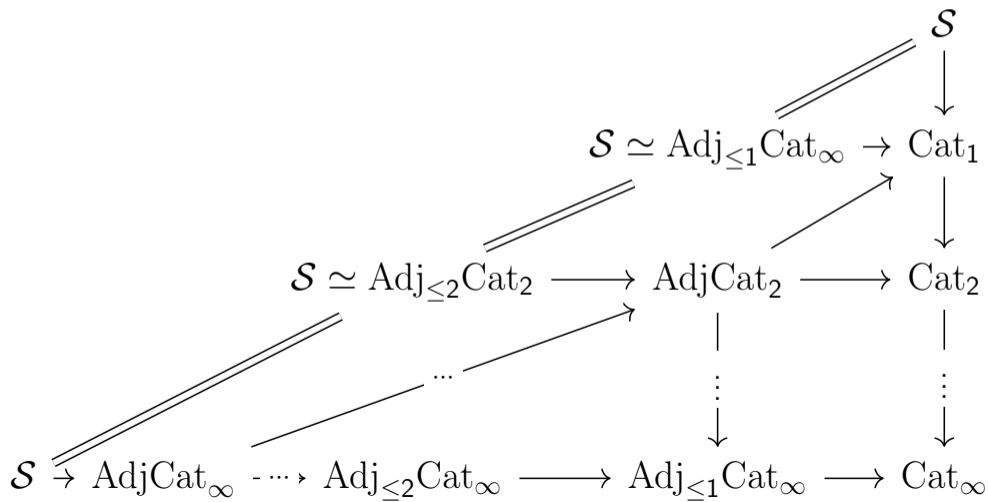
$$\sigma \mathcal{C} := 0 \xrightarrow{\mathcal{C}} 1$$

Ex: $\vec{\mathbb{D}}^n := \sigma^n \vec{\mathbb{D}}^0 \simeq *$

Dfn: a category has left adjoints for n -cells if it is $\sigma^{n-1}\ell$ -local.

$$\text{AdjCat}_\infty = \text{Cat}_\infty[\{\sigma^n \ell, \sigma^n r\}_{n \geq 0}^{-1}] \rightleftarrows \text{Cat}_\infty$$

Categories with all adjoints



Categories with all adjoints

Corollary

$$\begin{aligned}\text{AdjCat}_\infty &:= \text{Cat}_\infty[\{\sigma^n \ell, \sigma^n r\}_{n \geq 0}^{-1}] \\ &= \text{colim}(\text{AdjCat}_{n+1} \rightarrow \text{AdjCat}_n) \\ &= \text{colim}(\text{Adj}_{\leq n} \text{Cat}_\infty \rightarrow \text{Adj}_{\leq n-1} \text{Cat}_\infty)\end{aligned}$$

RMK: this is formal; could be $L\text{AdjCat}_\infty$ and $R\text{AdjCat}_\infty$ instead.

EX: monoidal ∞ -category with duals = connective category with adjoints.

RMK: the symmetric case needs CatSp .

Grothendieck constructions

A Grothendieck construction for $\mathcal{C} \in \mathcal{V}\text{Cat}$ is a fully faithful functor

$$\int_{\mathcal{C}} : \text{hom}_{\mathcal{V}\text{Cat}}(\mathcal{C}, \underline{\mathcal{V}\text{Cat}}) \hookrightarrow \mathcal{V}\text{Cat}/_{\mathcal{C}}.$$

They are usually obtained by pullback against a *universal fibration*.

$$\begin{array}{ccc} \int F & \longrightarrow & \underline{\mathcal{V}\text{Cat}}^{*//} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{C} & \xrightarrow{F} & \underline{\mathcal{V}\text{Cat}} \end{array}$$

$\mathcal{V}\text{Cat} =:$ $\text{Cat}_{(1,1)}$ (Grothendieck); CAT (Lurie); $\text{Cat}_{(2,2)}$ (Buckley); $(\infty, 2)$ (Abellán-Stern); (∞, n) (many authors); (∞, ∞) (Gepner–Heine, upcoming).

Grothendieck construction for ∞ -categories

Theorem (Gepner-Heine)

There is a universal fibration for Cat_∞ :

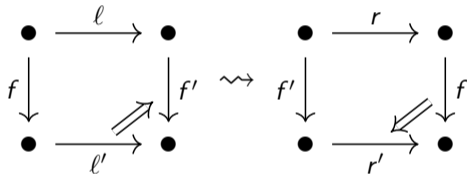
$$\begin{array}{ccc} \int F & \longrightarrow & \underline{\text{Cat}_\infty^{*\text{//}}^{\text{lax}}} \\ \downarrow & \lrcorner & \downarrow \\ \mathcal{C} & \xrightarrow{F} & \underline{\text{Cat}_\infty} \end{array}$$

DEF: $\text{Cocart}(\mathcal{C}) := \text{image of } \int_{\mathcal{C}} : \text{hom}_{\text{Cat}_\infty}(\mathcal{C}, \underline{\text{Cat}_\infty}) \hookrightarrow \text{Cat}_\infty/\mathcal{C}$

Lemma (mates correspondence)

$$\text{Fun}^{\text{lax}, L}(\mathcal{C}, \mathcal{D}) \simeq \text{Fun}^{\text{oplax}}(\mathcal{C}, \mathcal{D})^{\text{coop}}$$

Classically:



Lemma

$$\text{Fun}^{\text{lax}, L}(\mathcal{C}, \mathcal{D}) = \text{Fun}^{\text{lax}}(\mathcal{C}, \mathcal{D})^L$$

Proof strategy (?)

Lemma

For $\mathcal{C}, \mathcal{D} \in \text{AdjCat}_\infty$:

- $\text{Fun}^{\text{lax}}(\mathcal{C}, \mathcal{D}) \simeq \text{Fun}^{\text{oplax}}(\mathcal{C}, \mathcal{D})^{\text{coop}}$ (mates)
- $\text{Fun}^{\text{lax}}(\mathcal{C}, \mathcal{D}) \simeq \text{Fun}^{\text{oplax}}(\mathcal{C}, \mathcal{D})$ (further adjoint)

In AdjCat_∞ , this is simply $\text{Fun}^{\text{lax}}(\mathcal{C}, \mathcal{D}) \simeq \text{Fun}^{\text{oplax}}(\mathcal{C}, \mathcal{D})^{\text{coop}}$. In fact:

$$\text{Fun}^{\text{lax}}(\mathcal{C}, \mathcal{D}) \simeq \text{Fun}^{\text{oplax}}(\mathcal{C}, \mathcal{D})^{\text{coop}} \simeq \text{Fun}^{\text{lax}}(\mathcal{C}, \mathcal{D})^{\text{coop}} \simeq \text{Fun}^{\text{oplax}}(\mathcal{C}, \mathcal{D})$$

Related to the conjecture: $\otimes_{|\text{AdjCat}}$ is symmetric monoidal.

Proof strategy (???)

Recall the strategy for groupoids:

In any case, the Grothendieck construction of

$$\mathcal{C} \rightarrow \text{AdjCat}$$

is a “left-adj-fibration”, i.e. its fibers are in AdjCat , and it has cocartesian lifts. We see our failure: the cocartesian cells are those coming from equivalences, so this won't work out (in this model of (∞, ∞) -categories).

Either

- 1 move to the *coinductive* model of (∞, ∞) -categories
- 2 figure out some other Grothendieck construction (e.g. left Gray functors)

to-be-figured out! (this ~~Summer~~ Winter) stay tuned for updates

Valeu demais!