$\mathcal{V}(1, -) : \mathcal{V} \to \mathsf{Cat}$ is lax symmetric monoidal

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The first three definitions are coopted from [1].

Definition 1. Let $P \colon \mathcal{E} \to \mathcal{B}$ be a homomorphism of bicategories. A 1-cell $f \colon X \to Y$ in $\mathcal{E}$ is **Cocartesian** if:

(i) For every $g \colon X \to z$ in $\mathcal{E}$ with $h \colon Py \to Pz$ and isomorphism $\alpha \colon h \cdot Pf \Rightarrow Pg$, there exists $\hat{h} \colon y \to z$ and isomorphisms $\hat{\alpha} \colon \hat{h}f \Rightarrow g$, $\hat{\beta} \colon P\hat{h} \Rightarrow h$ satisfying the evident coherence.

(ii) 2-cells lift uniquely: given $\sigma \colon g \Rightarrow g'$ and compatible data (h, α), (h', α') with lifts $(\hat{h}, \hat{\alpha}, \hat{\beta})$, $(\hat{h}', \hat{\alpha}', \hat{\beta}')$, for any $\delta \colon h \Rightarrow h'$ compatible with σ, there exists a unique $\hat{\delta} \colon \hat{h} \Rightarrow \hat{h}'$ lifting δ coherently.

Definition 2. A 2-cell $\alpha \colon f \Rightarrow g \colon X \to Y$ in $\mathcal{E}$ is **Cocartesian** if it is Cocartesian as a morphism in the category $\mathcal{E}(X, Y)$.

Definition 3. A homomorphism $P \colon \mathcal{E} \to \mathcal{B}$ is a **Cocartesian 2-fibration** if:

(i) for any $e \in \mathcal{E}$ and $f \colon Pe \to b$, there exists a Cocartesian 1-cell $h \colon e \to a$ with $Ph = f$.

(ii) the components $P_{XY} : \mathcal{E}(X, Y) \to \mathcal{B}(PX, PY)$ are opfibrations.

(iii) the horizontal composite of any two Cocartesian 2-cells is Cocartesian.

Remark 4. All of Buckley's structural results for fibrations of bicategories (Propositions 3.1.8–3.1.16, Section 3.2) dualize to Cocartesian 2-fibrations (there called a 'coop fibration'). In particular, for any bicategory $\mathcal{B}$ there is a triaequivalence

$$\mathrm{el} \colon [\mathcal{B}, \mathsf{Bicat}] \xrightarrow{\simeq} \mathsf{CoCart}(\mathcal{B}).$$

Fact 5. If $P : \mathcal{E} \to \mathcal{B}$ is a Cocartesian fibration and $f : X \to Y$ is a morphism in $\mathcal{B}$, there is a well-defined homomorphism $f_! : \mathcal{E}_X \to \mathcal{E}_Y$ on fibers. The homomorphism sends $\widehat{X} \in \mathcal{E}_X$ to the codomain of a Cocartesian lift $\widehat{f} \colon \widehat{X} \to \widehat{Y}$, and a morphism $\widehat{X} \to \widetilde{X}$ in $\mathcal{E}_X$ is sent to a choice of lift depicted below:

$$\begin{array}{ccc} \widehat{X} & \longrightarrow & \widehat{Y} \\ {\scriptstyle g}\downarrow & \simeq & \downarrow{\scriptstyle g_!} \\ \widetilde{X} & \longrightarrow & \widetilde{Y} \end{array}$$

The rest of the axioms ensure that this assembles into a homomorphism.

Definition 6. We will consider a skeleton of the category of pointed finite sets and denote it by Γ^{op}. Objects are $\langle n\rangle = \{*, 1, \ldots, n\}$ for $n \geq 0$, and morphisms are basepoint-preserving maps. A morphism $\langle n\rangle \to \langle m\rangle$ is **inert** if the fiber over each non-basepoint is a singleton, and **active** if the fiber of the basepoint is a singleton.

Example: the maps $i\colon \langle n\rangle \to \langle 1\rangle$ sending $i \mapsto 1$ and $j \mapsto *$ for $j \neq i$ are inert.

Remark 7 (Fibrations over a category)*.* When the codomain of a homomorphism $P : \mathcal{E} \to \mathcal{B}$ is a 1-category and we ask if P is a Cocartesian 2-fibration, it suffices to find cartesian lifts for 1-morphisms: the hom-categories $\mathcal{B}(X, Y)$ are sets, so 'P is locally an opfibration' is automatic (every function is an opfibration), and so is 'closure of Cocartesian 2-cells under horizontal composition' (every 2-cell is an identity).

Definition 8. A **symmetric monoidal bicategory** is a Cocartesian 2-fibration $P\colon \mathcal{V} \to \Gamma$ with the following properties:

- Let $X \in \mathcal{V}$ be an object over $\langle n\rangle$ and consider the inert morphisms $i\colon \langle n\rangle \to \langle 1\rangle$, for $1 \leq i \leq n$. Using the fact, lift these to homomorphisms $\hat{\imath} : \mathcal{V}_{\langle n\rangle} \to \mathcal{V}_{\langle 1\rangle}$ which assemble into a homomorphism
$$(\hat{1}, \ldots, \hat{n})\colon \mathcal{V}_{\langle n\rangle} \to \mathcal{V}_{\langle 1\rangle} \times \cdots \times \mathcal{V}_{\langle 1\rangle}.$$
The condition is that this map is an equivalence of bicategories (for $n = 0$ we mean $\mathcal{V}_{\langle 0\rangle} \simeq *$).

Remark 9*.* Our whole strategy is to use the definition above to avoid dealing with tricategories.

Example: A *strict* symmetric monoidal 2-category $(\mathcal{W}, \otimes, \mathbb{1})$ defines a strict 2-functor $F\colon \Gamma^{\mathrm{op}} \to \mathsf{Bicat}$ sending $\langle n\rangle$ to $\mathcal{W}^{\times n}$, and a morphism $\rho\colon \langle n\rangle \to \langle m\rangle$ to the 2-functor $\mathcal{W}^{\times n} \to \mathcal{W}^{\times m}$ whose i-th component sends $(X_1, \ldots, X_n)$ to $\bigotimes_{j\in f^{-1}(i)} X_j$. In particular, if ρ is inert then $F(\rho)$ sends $(X_1, \ldots, X_n)$ to $(X_{\rho^{-1}(1)}, \ldots, X_{\rho^{-1}(n)})$.

This data can be encoded as a Cocartesian fibration $\mathcal{W}^{\otimes} \to \Gamma^{\mathrm{op}}$ by applying Buckley's Grothendieck construction. Objects of $\mathcal{W}^{\otimes}$ can be identified with lists $(X_1, \cdots, X_n)$ as objects of $\mathcal{W}$ with arbitrary length, and a morphism $(X_1, \cdots, X_n) \to (Y_1, \cdots, Y_m)$ consists of a pair (ρ, f), where

- $\rho\colon \langle n\rangle \to \langle m\rangle$ is a morphism in Γ^{op}.
- $f\colon (\bigotimes_{f(i)=1} X_i, \ldots, \bigotimes_{f(j)=m} X_i) \to (Y_1, \cdots, Y_m)$ is a list of 2-functors between each entry.

In particular, a morphism (ρ, f) is Cocartesian iff f is an equivalence. Moreover, if ρ is inert then a Cocartesian morphism $(\rho, f)\colon (X_1, \ldots, X_n) \to (Y_1, \ldots, Y_m)$ is a list of equivalences $X_{\rho^{-1}(i)} \simeq Y_i$.

Example : $\mathsf{Cat}^{\times} \to \Gamma^{\mathrm{op}}$ is the Cocartesian fibration encoding the strict symmetric monoidal 2-category of categories, functors, and natural transformations, equipped with the categorical product.

Definition 10. A **lax symmetric monoidal functor** between symmetric monoidal bicategories $P\colon \mathcal{V} \to \Gamma$ and $Q\colon \mathcal{W} \to \Gamma$ is a homomorphism $F\colon \mathcal{V} \to \mathcal{W}$ over Γ that preserves Cocartesian 1-cells over inert morphisms.

Definition 11. A **unit** of a symmetric monoidal bicategory $\mathcal{V} \to \Gamma$ is an object $\mathbb{1}$ over $\langle 0 \rangle$.

The following construction will be relatively long, but it will trivialize the proof of the main theorem of this note. This is somewhat reinforcing the idea that the lax monoidal *data* is being repackaged as the Cocartesianess *property* of the fibrations involved.

Construction 12. Let $\mathcal{V} \to \Gamma$ be a symmetric monoidal bicategory and choose:

(a) for each $n \geq 1$, a lift of each inert morphism $i\colon \langle n \rangle \to \langle 1 \rangle$ to a homomorphism on the fibers $\hat{\imath}\colon \mathcal{V}_{\langle n \rangle} \to \mathcal{V}_{\langle 1 \rangle}$. We will write X_i for $\hat{\imath}(X)$.

(b) for each $n \geq 1$, a lift of the unique active morphism $\langle n \rangle \to \langle 1 \rangle$ to a homomorphism on the fibers. Identifying X with $(X_1, \ldots, X_n)$, we will write $\bigotimes X_i$ for the image of X.

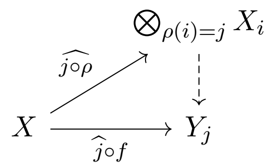
(c) a unit $\mathbb{1} \in \mathcal{V}_{\langle 0 \rangle}$, and for each $n \geq 1$ a lift of the unique morphism $\langle 0 \rangle \to \langle n \rangle$ to a homomorphism on the fibers. We will write $(\mathbb{1}, \ldots, \mathbb{1}) \in \mathcal{V}_{\langle n \rangle}$ for the image of $\mathbb{1}$, and $\mathbb{1}^{\otimes n}$ for its further image under the (active) homomorphism $\mathcal{V}_{\langle n \rangle} \to \mathcal{V}_{\langle 1 \rangle}$ in (b); note that $\mathbb{1}^{\otimes n} \simeq \mathbb{1}$.

Define a homomorphism $\mathcal{V}(\mathbb{1}, -) : \mathcal{V} \to \mathsf{Cat}^{\times}$ over Γ by sending

- an object $X \in \mathcal{V}_{\langle n \rangle}$ to $(\mathcal{V}_{\langle 1 \rangle}(\mathbb{1}, X_1), \ldots, \mathcal{V}_{\langle 1 \rangle}(\mathbb{1}, X_n)) \in \mathsf{Cat}^{\times}_{\langle n \rangle}$.
- a morphism $f\colon X \to Y$, over $\rho\colon \langle n \rangle \to \langle m \rangle$, to the morphism with components

$$\begin{aligned}
\prod_{\rho(i)=j} \mathcal{V}_{\langle 1 \rangle}(\mathbb{1}, X_i) &\xrightarrow{\sim} \mathcal{V}_{\langle k \rangle}((\mathbb{1}, \ldots, \mathbb{1}), (X_i)_{\rho(i)=j}) \qquad \text{where } k\colon = \rho^{-1}(j) \\
&\to \mathcal{V}_{\langle 1 \rangle}(\mathbb{1}^{\otimes k}, \bigotimes_{\rho(i)=j} X_i) \\
&\to \mathcal{V}_{\langle 1 \rangle}(\mathbb{1}, \bigotimes_{\rho(i)=j} X_i) \\
&\to \mathcal{V}_{\langle 1 \rangle}(\mathbb{1}, Y_j).
\end{aligned}$$

  Where the first map is the equivalence $\mathcal{V}_{\langle n \rangle} \simeq \mathcal{V}_{\langle 1 \rangle}^{\times k}$, the second is induced by the active map $\langle k \rangle \to \langle 1 \rangle$, the third is using $\mathbb{1}^{\otimes n} \simeq \mathbb{1}$, and the last one is obtained by precomposition with the following morphism: consider the morphism $\hat{\jmath} \circ f : X \to Y_j$ over $j \circ \rho$, and take a Cocartesian lift $\widehat{j \circ \rho}$, whose domain can be identified with $\bigotimes_{\rho(i)=j} X_i$ by definition; the morphism $\bigotimes_{\rho(i)=j} X_i \to Y_j$ is the following dashed arrow:



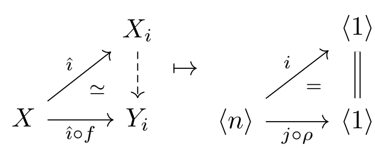
The action on 2-cells is obtained by using Cocartesianess of $X \to \bigotimes_{\rho(i)=j} X_i$; the compositor and unitor isomorphisms are obtained by comparing Cocartesian lifts of the same base morphism, and Cocartesianess yields coherence. However, since the main theorem below doesn't rely on these details (only that they assemble into a well-defined homorphism), we will omit them.

Theorem 13

The functor $\mathcal{V}(\mathbb{1}, -)\colon \mathcal{V} \to \mathsf{Cat}^{\times}$ is lax symmetric monoidal.

Proof: Since $\mathcal{V}(\mathbb{1}, -)$ is a homomorphism over Γ by construction, we only have to check that Cocartesian 1-cells over inert morphisms are preserved.

On an inert morphism $\rho : \langle n \rangle \to \langle m \rangle$, the set $\rho^{-1}(j)$ has a single elements, so the first three maps in the chain of morphisms above are equivalences, and for the last one it suffices to show that $X_i \to Y_i$ is an equivalence. This morphism was obtained as the dashed arrow below:



Since $\hat{\imath}$, $\hat{\jmath}$ and f are Cocartesian by hypothesis, the proof of [1, Proposition 3.1.15] shows that the dashed arrow we are looking at is an equivalence. Cocartesian maps in $\mathsf{Cat}^{\times}$ are equivalences, so we are done. □

Remark 14. The functor is only 'lax' because it does not preserve Cocartesian morphisms in general, and in particular it does not preserve active morphisms. For instance a Cocartesian lift of an active morphism $\langle 2 \rangle \to \langle 1 \rangle$ sends $X \simeq (X_1, X_2)$

$$\mathcal{V}_{\langle 1 \rangle}(\mathbb{1}, X_1) \times \mathcal{V}_{\langle 1 \rangle}(\mathbb{1}, X_2) \simeq \mathcal{V}_{\langle 2 \rangle}((\mathbb{1}, \mathbb{1}), (X_1, X_2)) \to \mathcal{V}_{\langle 1 \rangle}(\mathbb{1}, X_1 \otimes X_2),$$

and this map is not in general an equivalence.

Disclaimer 15. A stub for this document was produced by Claude AI via a prompt detailing the structure and order of the contents of this note, which subsequently had most of its words changed.

References

[1] M. Buckley, **Fibred 2-categories and bicategories**, J. Pure Appl. Algebra **218**(6), 2014, 1034–1074.