$\mathbf{Pshv}(\mathbf{C}) \overset{L}{\otimes} \mathbf{Pshv}(\mathbf{D}) \simeq \mathbf{Pshv}(\mathbf{C} \times \mathbf{D})$

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Notation 1. All categories in this note are presentable.

- The symbol $[\mathcal{C} \xrightarrow{L} \mathcal{D}]$ denotes the category of colimit preserving functors $\mathcal{C} \to \mathcal{D}$.
- The symbol $[\mathcal{C} \times \mathcal{D} \xrightarrow{LL} \mathcal{E}]$ denotes the category of functors preserving colimits in each entry.
- The symbol $[\mathcal{C} \to \mathcal{D}]$ denotes the category of all functors $\mathcal{C} \to \mathcal{D}$.

Definition 2. The tensor product $\mathcal{C} \overset{L}{\otimes} \mathcal{D}$ is defined by the universal property

$$[\mathcal{C} \overset{L}{\otimes} \mathcal{D} \xrightarrow{L} \mathcal{E}] \simeq [\mathcal{C} \times \mathcal{D} \xrightarrow{LL} \mathcal{E}].$$

Fact 3. The tensor product is closed via $[\mathcal{C} \overset{L}{\otimes} \mathcal{D} \xrightarrow{L} \mathcal{E}] \simeq [\mathcal{C} \xrightarrow{L} [\mathcal{D} \xrightarrow{L} \mathcal{E}]]$.

Fact 4. The category of presheaves of $\mathcal{C}$ is its free cocompletion, i.e. $[\mathbf{Pshv}(\mathbf{C}) \xrightarrow{L} \mathcal{E}] \simeq [\mathcal{C} \to \mathcal{E}]$.

Theorem 5

There is an equivalence $\mathbf{Pshv}(\mathbf{C}) \overset{L}{\otimes} \mathbf{Pshv}(\mathbf{D}) \simeq \mathbf{Pshv}(\mathbf{C} \times \mathbf{D})$.

Proof: For every $\mathcal{E}$ we have the equivalences

$$\begin{aligned}
[\mathbf{Pshv}(\mathcal{C}) \overset{L}{\otimes} \mathbf{Pshv}(\mathcal{D}) \xrightarrow{L} \mathcal{E}] &\simeq [\mathbf{Pshv}(\mathcal{C}) \xrightarrow{L} [\mathbf{Pshv}(\mathcal{D}) \xrightarrow{L} \mathcal{E}]] \\
&\simeq [\mathbf{Pshv}(\mathcal{C}) \xrightarrow{L} [\mathcal{D} \to \mathcal{E}]] \\
&\simeq [\mathcal{C} \to [\mathcal{D} \to \mathcal{E}]] \\
&\simeq [\mathcal{C} \times \mathcal{D} \to \mathcal{E}] \\
&\simeq [\mathbf{Pshv}(\mathcal{C} \times \mathcal{D}) \xrightarrow{L} \mathcal{E}],
\end{aligned}$$

whence $\mathbf{Pshv}(\mathcal{C}) \overset{L}{\otimes} \mathbf{Pshv}(\mathcal{D}) \cong \mathbf{Pshv}(\mathcal{C} \times \mathcal{D})$ because they represent the same functor. ■