Flagged categories present valent categories

Daniel Teixeira

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Let $\mathcal{S}$ be the $(\infty, 1)$-category of spaces, and recall the Segal maps: $\text{ɬ}_1 \times_{\text{ɬ}_0} \cdots \times_{\text{ɬ}_0} \text{ɬ}_1 \to \text{ɬ}_n$.

Definition 1. A **category** is a presheaf $\mathcal{C} \in \mathcal{S}^{\Delta^{\mathrm{op}}}$ local with respect to the Segal maps.

Categories form a reflective category $\mathbf{Cat}_{\mathrm{valent}} \hookrightarrow \mathcal{S}^{\Delta^{\mathrm{op}}}$. They can be pictured like this:

$$\begin{array}{ccccc} & & \mathrm{eq}(\mathcal{C}) & & \\ & \nearrow & \downarrow & & \\ \mathrm{ob}(\mathcal{C}) & \rightleftarrows & \mathrm{mor}(\mathcal{C}) & \longleftarrow & \mathrm{mor}(\mathcal{C}) \times_{\mathrm{ob}(\mathcal{C})} \mathrm{mor}(\mathcal{C}). \end{array}$$

Here $\mathrm{eq}(\mathcal{C})$, which we will define as $\mathrm{Map}(J, \mathcal{C})$ via the Fact below.

Example : the terminal category $\top$ is defined by $\mathrm{ob}(\top) = \mathrm{mor}(\top) = *$.

Example : the **walking isomorphism** is the category J with $\mathrm{ob}(J) = \{0, 1\}$ and $\mathrm{mor}(J) = \{f, f^{-1}\} \sqcup \{0, 1\}$, where $f : 0 \to 1 : f^{-1}$.

Fact 2. The functors $\mathcal{C} \mapsto \mathrm{eq}(\mathcal{C})$ and $\mathcal{C} \mapsto \mathrm{ob}(\mathcal{C})$ are represented by J and $\top$. The map $\mathrm{ob}(\mathcal{C}) \to \mathrm{eq}(\mathcal{C})$ above is given by precomposing with $J \xrightarrow{!} \top$.

Definition 3. A category is **univalent** if the map $\mathrm{ob}(\mathcal{C}) \to \mathrm{eq}(\mathcal{C})$ is an equivalence.

This definition immediately makes $\mathbf{Cat}_{\mathrm{univalent}}$ the reflective subcategory of $\mathbf{Cat}_{\mathrm{valent}}$ at the $\{J \xrightarrow{!} \top\}$-local objects. Hence, unless indicated, categories are not assumed to be univalent. We will sometimes emphasize this by referring to not-necessarily-univalent categories as **valent categories**.

Example : the point is univalent and the walking isomorphism is not.

Example : a *strict category* is a valent category in which $\mathrm{ob}(\mathcal{C})$ is a set; it is univalent iff it is gaunt.

Example : every space gives an univalent category by using identities everywhere.

Remark 4. In terms of models, we could write the following table:

concept	$\mathcal{S}$	valent categories	univalent categories
model	$\mathbf{sSet}_{\mathrm{Quillen}}$	Segal spaces	complete Segal spaces

The model category of Segal spaces is the left Bousfield localization of the Reedy model structure of $\mathbf{sSet}^{\Delta^{\mathrm{op}}}_{\mathrm{Quillen}}$ at the Segal maps. The model structure for complete Segal spaces is the further localization at the map $N\mathbf{Iso} \to *$, where $N\mathbf{Iso}$ is the nerve of the walking isomorphism (i.e. the category $f : 0 \leftrightarrows 1 : f^{-1}$ with $ff^{-1} = \mathrm{id}_1$ and $f^{-1}f = \mathrm{id}_0$), seen as a bisimplicial set. These models are simplicial so they live in the Bergner model structre for $(\infty, 1)$-categories.

Definition 5. A **flagged category** is a functor $\mathcal{G} \to \mathcal{C}$, where $\mathcal{G}$ is a space and $\mathcal{C}$ is univalent, and such that the induced map $\pi_0(\mathcal{G}) \to \pi_0(\mathrm{ob}(\mathcal{C}))$ is surjective.

We will say that "$\mathcal{C}$ is flagged by $\mathcal{G}$" when referring to a flagged category.

Example : the category of rings and bimodules is flagged by the groupoids of rings and ring isomorphisms.

Example : the category of finite dimensional Hilbert spaces and linear maps is flagged by the groupoid of Hilbert spaces and unitary isomorphisms.

A flagged category $F\colon \mathcal{G} \to \mathcal{C}$ yields a valent category $u\mathcal{C}$ by setting $\mathrm{ob}(u\mathcal{C}) := \mathcal{G}$ and

$$\mathrm{mor}(u\mathcal{C}) := (\mathcal{G} \times \mathcal{G}) \times_{\mathrm{ob}(\mathcal{C})} \mathrm{mor}(\mathcal{C}),$$

where the pullback is along F and the source and target maps. This construction yields a functor $u\colon \mathbf{FlCat} \to \mathbf{Cat}_{\mathrm{valent}}$, and the following is a theorem of Ayala-Francis: [1]

Theorem 6

$u : \mathbf{FlCat} \to \mathbf{Cat}_{\mathrm{valent}}$ is an equivalence.

Sketch of proof: A functor $\mathcal{S}^{\Delta^{\mathrm{op}}} \to \mathbf{FlCat}$ is obtained by sending $X_\bullet$ to the flagged category $X_0 \to X$, where X is the univalent category which essentially looks like $X_0 \rightleftarrows X_1 \longleftarrow X_1 \times_{X_0} X_1 \cdots$. This functor localizes exactly the Segal maps, so $\mathbf{FlCat} \simeq \mathbf{Cat}_{\mathrm{valent}}$; u gives a particular realization of this equivalence. □

The point of this note is to emphasize that flagged categories are tools to present valent categories:

Example : the category of rings and bimodules, flagged by ring isomorphisms, presents a valent category with: rings and isomorphisms as the space of objects, and rings and Morita equivalences as the space of equivalences. It is not univalent because Morita equivalences are coarser than ring isomorphisms.

Example : the category of f. d. Hilbert spaces, flagged by unitary isomorphisms, presents a valent category with: Hilbert spaces and unitary isomorphisms as the space of objects, and Hilbert spaces and invertible linear maps as the space of morphisms. It is not univalent because $\mathrm{U}(n) \neq \mathrm{GL}(n, \mathbb{C})$.

Remark 7. One might expect valent $(1, 1)$-categories to be presented by flaggings $\mathcal{G} \to \mathcal{C}$ where $\mathcal{G}$ is a 1-groupoid and $\mathcal{C}$ is a $(1, 1)$-univalent category. This is not true: univalent $(1, 1)$-categories are gaunt, so they cannot present the examples above, even though they are $(1, 1)$-categories.

Remark 8. This yoga extends to n-categories: a flagged n-category is a sequence $\mathcal{C}_0 \to \mathcal{C}_1 \to \cdots \to \mathcal{C}_n$ of univalent k-categories with each functor is as surjective as possible, and flagged n-categories present valent n-categories.

[1] Flagged higher categories, in: Topology and Quantum Theory in Interaction, Contemporary Mathematics 718, AMS (2018) [arXiv:1801.08973, ISBN:978-1-4704-4941-4]