Review

Enriched over n-Cat, taking values in S.

(cf. 2.6 Cartesian enrichment)

$$0\text{-Cat} := S$$

$$(n+1)\text{-Cat} := \text{Cat}(n\text{-Cat}, S)$$

$$\infty\text{-Cat} := \text{Lim}^{\text{Pr}^R}(\cdots \to n\text{Cat} \to \cdots \to S)$$

$$= \text{Colim}^{\text{Pr}^L}(S \to \text{Cat} \to \cdots)$$

$$\boxed{\mathcal{C} = \underset{n}{\text{colim}}\, L^n(\mathcal{C}) \quad \in \infty\text{-Cat}}$$

$$0\text{-Cat}^{\text{str}} := \text{Set}$$

$$(n+1)\text{-Cat}^{\text{str}} := \text{Cat}(n\text{-Cat}^{\text{str}}, \text{Set})$$

$$\infty\text{-Cat}^{\text{str}} := \underset{n}{\text{Lim}}^{\text{Pr}^R}(n\text{-Cat}^{\text{str}})$$

$$0\text{-Cat}^{\text{univ}} := S$$

$$(n+1)\text{-Cat}^{\text{univ}} := \text{Cat}^{\boxed{\text{univ}}}(n\text{-Cat}^{\text{univ}}, S)$$

the underlying category

$$\infty\text{-Cat}^{\text{univ}} := \underset{n}{\text{Lim}}^{\text{Pr}^R}(n\text{-Cat}^{\text{univ}})$$

univalence

$$\begin{tikzcd} \{\bullet \simeq \bullet\} \arrow[r] \arrow[d] & \mathcal{C} \\ \bullet \arrow[ur, dashed, "\exists"'] \arrow[u, phantom, shift left=6, "\simeq"] & \end{tikzcd}$$

Remark

$\mathcal{V}$: presentable cartesian mon.

$$\begin{tikzcd} \mathcal{V} \arrow[r, shift right, "\mathcal{V}(1,-)"'] \arrow[r, phantom, "\perp"] & S \arrow[l, shift right, "-\cdot 1"'] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \text{Cat}(\mathcal{V}, S) \arrow[r, shift right] \arrow[r, phantom, "\perp"] & \text{Cat}(S) \arrow[l, shift right, "\Delta"'] \end{tikzcd}$$

The univalence on the underlying cat.

$$\Longleftrightarrow \quad \begin{tikzcd} \Delta\{\bullet \simeq \bullet\} \arrow[r] \arrow[d] & \mathcal{C} \\ \Delta\bullet \arrow[ur, dashed, "\exists"'] \arrow[u, phantom, shift left=6, "\simeq"] & \end{tikzcd}$$

$$\begin{tikzcd} \infty\text{-Cat}^{\text{gaunt}} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \infty\text{-Cat}^{\text{str}} \arrow[r] \arrow[d] & \cdots & 0\text{-Cat}^{\text{gaunt}} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & 0\text{-Cat}^{\text{str}} \arrow[d] \\ \infty\text{-Cat}^{\text{univ}} \arrow[r] & \infty\text{-Cat} & \cdots \arrow[r] & 0\text{-Cat}^{\text{univ}} \arrow[r] & 0\text{-Cat} \end{tikzcd}$$

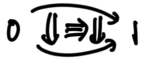
Prop [Prop 3.2.13] $\mathcal{C}$ is gaunt ∞-caty. $\Longleftrightarrow$ ${}^{\forall}n$ $L_n(\mathcal{C})$ is (n,n).

The two constructions

- Suspension

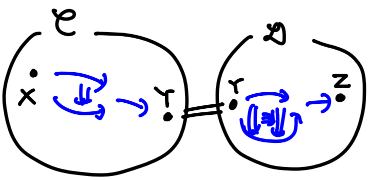
$\mathcal{C}$: ∞-cat. $\rightsquigarrow$ $S(\mathcal{C})$: its suspension

$$\left(\begin{tikzcd} & * \arrow[dr] \arrow[dd, phantom, "\Downarrow"] & \\ \mathcal{C} \arrow[ur] \arrow[dr] & & S(\mathcal{C}) \\ & * \arrow[ur] & \end{tikzcd} \quad : \text{lax cocomma}\right)$$

n	0	1	2	3
$\mathbb{D}^n = S^n(*)$ $\uparrow$	$*$	$0 \xrightarrow{*} 1$		
$\partial\mathbb{D}^n = S^n(\emptyset)$	$\emptyset$	$0 \quad 1$		

- Bipointed wedge

$$\left(\begin{matrix}\partial\mathbb{D}_1 \xrightarrow{X,Y} \mathcal{C}, \\ \partial\mathbb{D}_1 \xrightarrow{Y,Z} \mathcal{D}\end{matrix}\right) \rightsquigarrow \partial\mathbb{D}_1 \xrightarrow{X,Z} \mathcal{C} \amalg_Y \mathcal{D}$$



$$\left(\begin{tikzcd} * \arrow[dr, "X"'] & & * \arrow[dl, "Y"'] \arrow[dr, "Y"] & & * \arrow[dl, "Z"] \\ & \mathcal{C} \arrow[dr] & \overset{\simeq}{\wedge} & \mathcal{D} \arrow[dl] & \\ & & \mathcal{C} \amalg_Y \mathcal{D} & & \end{tikzcd} \quad : \text{pushout}\right)$$

Def . $\Theta' \subseteq \partial\mathbb{D}^1/\infty\text{-Cat}$ is the smallest full sub caty

that
- includes $\partial\mathbb{D}^1 \to \mathbb{D}^0$
- closed under suspension
- " " bipointed wedge

$$\begin{tikzcd} * \arrow[dr] & & * \arrow[dl] & & & \mathcal{C} \arrow[dl] \arrow[dr] & \\ & \mathcal{C} & & \rightsquigarrow & * \arrow[dr] \arrow[rr, phantom, "\Rightarrow"] & & * \arrow[dl] \\ & & & & & S(\mathcal{C}) & \end{tikzcd}$$

.

$$\begin{tikzcd} \partial\mathbb{D}^1/\infty\text{-Cat} \arrow[r] & \infty\text{-Cat} \\ \Theta' \arrow[u, phantom, sloped, "\subseteq"] \arrow[r, mapsto, "\mathrm{Im}"] & {:\ \Theta} \arrow[u, phantom, sloped, "\subseteq"] \end{tikzcd}$$

Remark $\Theta_0 \hookrightarrow \Theta_1 \hookrightarrow \Theta_2 \hookrightarrow \cdots$

Θ_n : the full sub of (n,n)-categories

$\Longleftrightarrow$ those categories obtained with n-times of suspensions

Thm

$$\begin{tikzcd}
\underline{\infty\text{-Cat}} \arrow[r, hook, bend right, "N"'] \arrow[r, phantom, "\perp"] & \underline{\mathrm{Fun}(\Theta^{op}, S)} \arrow[l, dashed, bend right, "R\ \text{fin.prod.pres.}"'] \\
\underline{n\text{-Cat}} \arrow[u, hook] \arrow[r, hook, bend right] \arrow[r, phantom, "\perp"] & \underline{\mathrm{Fun}(\Theta_n^{op}, S)} \arrow[u, hook] \arrow[l, bend right]
\end{tikzcd}
,\quad
\begin{tikzcd}
\underline{\infty\text{-Cat}^{str}} \arrow[r, hook, bend right, "N"'] \arrow[r, phantom, "\perp"] & \underline{\mathrm{Fun}(\Theta^{op}, \mathrm{Set})} \arrow[l, dashed, bend right, "R\ \text{fin.pro.pres.}"'] \\
\underline{n\text{-Cat}^{str}} \arrow[u, hook] \arrow[r, hook, bend right] \arrow[r, phantom, "\perp"] & \underline{\mathrm{Fun}(\Theta_n^{op}, \mathrm{Set})} \arrow[u, hook] \arrow[l, bend right]
\end{tikzcd}$$

proof (might not work in ∞-setting.)

$\{D_0, D_1, \dots, D_n\}$ are generating in $\underline{n\text{-Cat}}$,

so the fin. colim. closure is dense in $\underline{n\text{-Cat}}$.

Θ_n contains this, so it is dense. (Finite products preserving : omitted)

$$\begin{tikzcd}
\underline{\mathrm{Fun}(\Theta_n^{op}, S)} \arrow[d, bend right] \arrow[d, phantom, "\dashv"] \arrow[r, hook, bend right] \arrow[r, phantom, "\top"] & \cdots \arrow[l, bend right] \arrow[d, bend right] \arrow[d, phantom, "\dashv"] & \underline{\mathrm{Fun}(\Theta^{op}, S)} \arrow[d, bend right, "L"'] \arrow[d, phantom, "\dashv"] & = \lim\limits_n \underline{\mathrm{Fun}(\Theta_n^{op}, S)} \\
\underline{n\text{-Cat}} \arrow[u, hook, bend right] \arrow[r, hook, bend right] \arrow[r, phantom, "\top"] \arrow[ur, phantom, "(BC)"] & \cdots \arrow[l, bend right] \arrow[u, hook, bend right] & \underline{\infty\text{-Cat}} \arrow[u, hook, bend right, "i"'] & = \lim\limits_n \underline{n\text{-Cat}}
\end{tikzcd}$$

$$\begin{tikzcd}
\underline{\Theta_n} \arrow[r, "J_n"] \arrow[d, hook, "i_n"'] & \underline{\Theta_{n+1}} \arrow[d, hook, "i_{n+1}"] \\
\underline{n\mathrm{Cat}} \arrow[r, "I_n"'] & (n+1)\mathrm{Cat}
\end{tikzcd}$$

$$\begin{tikzcd}
\cdot \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\simeq"] & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}$$
because

$$\mathrm{Colim}^{(\mathrm{Lan}_{J_n}F)-} i_{n+1}- \cong \mathrm{Colim}^{\mathrm{Colim}^{\Theta_{n+1}(-,J_n*)}F*} i_{n+1}-$$

$$\cong \mathrm{Colim}^{\mathrm{Colim}^{F*}\Theta_{n+1}(-,J_n*)} i_{n+1}-$$

$$\cong \mathrm{Colim}^{F*}\, \mathrm{Colim}^{\Theta_{n+1}(-,J_n*)} i_{n+1}-$$

$$\cong \mathrm{Colim}^{F*}\, i_{n+1}J_n(*) \cong I_n(\mathrm{Colim}^F i_n)$$

$$\begin{tikzcd}
\Theta_n^{op} \arrow[r, "F"] \arrow[d, "J_n"'] & S \arrow[dl, phantom, "\Downarrow"] \\
\Theta_{n+1}^{op} \arrow[ur, dashed, "\mathrm{Lan}_{J_n}F"'] &
\end{tikzcd}$$

i: f.f. because it is induced by f.f. functors.

L preserves fin. products ~ by fin.prod.pres. functors.

Steiner ∞-categories

Reference

- Ara, Gagna, Ozornova, Rovelli. "A categorical characterization of strong Steiner ω-categories"
- Steiner's original paper

Def $\mathcal{C}$: str. ∞-caty is a polygraph (ically generated)

if $\exists$ $\mathcal{E}_n \subseteq \{$n-cells in $\mathcal{C}\}$ $\quad$ $f \in \mathcal{E}_n$ are called atomic

$$\begin{tikzcd}
 & \emptyset \arrow[r] \arrow[dl] & \coprod_{\mathcal{E}_0} \mathbb{D}^0 \arrow[d] & \coprod_{\mathcal{E}_1} \partial\mathbb{D}^1 \arrow[r] \arrow[dl] & \coprod_{\mathcal{E}_1} \mathbb{D}^1 \arrow[d] & & \coprod_{\mathcal{E}_n} \partial\mathbb{D}^n \arrow[r] \arrow[d] & \coprod_{\mathcal{E}_n} \mathbb{D}^n \arrow[d] & \\
\emptyset \arrow[rr] & & L_0(\mathcal{C}) \arrow[rr] \arrow[ul, phantom, "\ulcorner", very near start] & & L_1(\mathcal{C}) \arrow[ul, phantom, "\ulcorner", very near start] & {} \arrow[r] & L_{n-1}(\mathcal{C}) \arrow[r] & L_n(\mathcal{C}) \arrow[ul, phantom, "\ulcorner", very near start] & \cdots
\end{tikzcd}$$

$$\mathcal{E}_0 \leftleftarrows \mathcal{E}_1 \leftleftarrows \mathcal{E}_2 \cdots \leftleftarrows \mathcal{E}_n \leftleftarrows \cdots$$ is a globular set.

Remark

$$\begin{tikzcd}
\omega\text{-Cat} \arrow[r, shift right, "\perp"'] \arrow[r, phantom, "\perp"] & {[\mathbb{G}^{op}, \mathrm{Set}]} \arrow[l, shift right] \\
 & \mathbb{G} \arrow[ul, "\mathbb{D}"] \arrow[u, "よ"']
\end{tikzcd}$$

is monadic.

$\infty\text{-Cat}^{\text{str.}}$ $\not\simeq$ ω-Cat

$[\mathbb{G}^{op}, \mathrm{Set}]$: the category of globular sets.

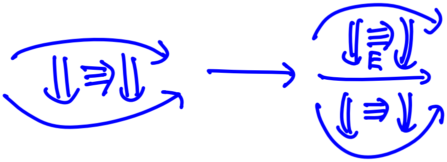
$\mathbb{D}_0 \rightrightarrows \mathbb{D}_1 \rightrightarrows \mathbb{D}_2 \rightrightarrows \cdots$ in ω-Cat

There are more polygraphs than free algebras for this.

Notation $\mathcal{C} = \langle \mathcal{E} \rangle = \langle \mathcal{E}_n \rangle_n$

Def $\mathcal{C} = \langle \mathcal{E} \rangle$, $\quad X$: m-cell in $\mathcal{C}$

$\mathrm{supp}(X) := \{ E \in \mathcal{E}_m \mid E$ is a factor in $X \}$

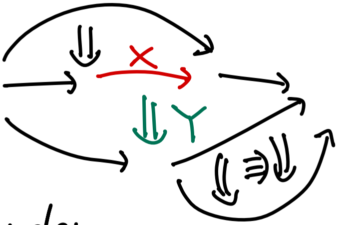
$X =$ 

$\leq_{\mathbb{N}}$ is the preorder generated by

$$X \leq'_{\mathbb{N}} Y \overset{\text{def}}{\iff} \begin{cases} X \in \operatorname{supp}(s_m(Y)) \\ \text{or } Y \in \operatorname{supp}(t_n(X)) \end{cases}$$

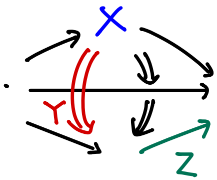
$(X \in \mathcal{E}_m, Y \in \mathcal{E}_n)$

Example $m = 1, n = 2$



Remark $\leq'_{\mathbb{N}}$ itself is not a preorder.

$X_0 \leq_{\mathbb{N}} Y_2 \leq_{\mathbb{N}} Z_1$



Def $\mathcal{C} = \langle \mathcal{E} \rangle$

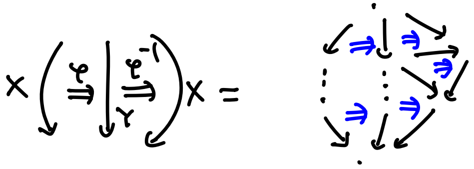
$\mathcal{E}$ is strongly loop-free $\overset{\text{def}}{\iff}$ $\leq_{\mathbb{N}}$ is antisymmetric

$$\overset{\text{def}}{\iff} \left(\underset{(n \in \mathbb{N})}{X_0 \leq'_{\mathbb{N}} X_1 \leq' \cdots \leq' X_n = X_0} \right) \Rightarrow n = 0$$

A Steiner category is a polygraph $\langle \mathcal{E} \rangle$ with a strongly loop free $\mathcal{E}$.

Remark Steiner is gaunt.

Proof sketch X, Y : 1-cells in $\mathcal{C}$



(all of them are atomic)

However, none of them should exist!

Stuff about colimits are omitted.

Examples

- Objects in Θ are Steiner categories.

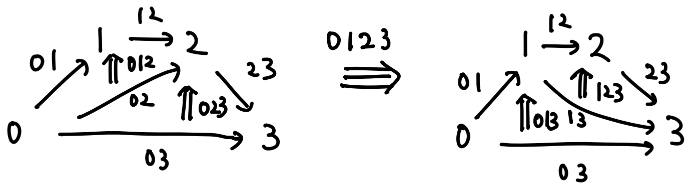
$\Leftarrow$
- $*$ is Steiner.
- Steiner categories are closed under bipointed wedge.

  $\left(X \leq'_{\mathbb{N}} Y \text{ in } \mathcal{C} \amalg_{Y} \mathcal{D} \text{ implies either } \begin{array}{l} X, Y \in \mathcal{C}, \\ X, Y \in \mathcal{D}, \\ X \in \mathcal{C}, Y \in \mathcal{D} \end{array} \right)$
- Steiner categories are closed under suspension.

  $\left(\begin{array}{l} \text{For } X \in \mathcal{C} \\ 0 \leq'_{\mathbb{N}} X \leq'_{\mathbb{N}} 1 \ \ \& \ \ 1 \nleq_{\mathbb{N}} X \nleq_{\mathbb{N}} 0 \end{array} \right)$

- Orientals are Steiner categories

Δ_3 :



$\leq_\pi$: (0-cells) $0 \leq 1 \leq 2 \leq 3$

(1-cells)

$$\begin{tikzcd}
 & & 01 \\
 & 02 \arrow[ur, phantom, sloped, "\leq"] \arrow[dr, phantom, sloped, "\leq"] & \\
03 \arrow[ur, phantom, sloped, "\leq"] \arrow[dr, phantom, sloped, "\leq"] & & 12 \\
 & 13 \arrow[dr, phantom, sloped, "\leq"] & \\
 & & 23
\end{tikzcd}$$

(2-cells)

$023 \leq 012$

$013 \leq 123$

Steiner theory

- $\underline{\mathbb{N}}$: discrete (Ab-) cat.

- $\underline{ch}$: $\underline{cMon\text{-}Cat}$

$$obj(\underline{ch}) = \mathbb{N}, \quad \underline{ch}(m,n) = \begin{cases} \mathbb{N} & (m=n,n+1) \\ 0 & (\text{otherwise}) \end{cases}$$

$$\underline{Ch_{\mathbb{Z}}} = \underline{cMon\text{-}Cat}(\underline{ch}, \underline{Ab})$$

- $\underline{dch}$: Ab-cat

$$obj(\underline{dch}) = \underset{\substack{\rotatebox{90}{$\in$} \\ n,}}{\mathbb{N}} \amalg \underset{\substack{\rotatebox{90}{$\in$} \\ m^+}}{\mathbb{N}}, \quad \begin{aligned} &\underline{dch}(m,n) = \underline{ch}(m,n) \\ &\underline{dch}(m^+,n) = 0 \\ &\underline{dch}(m,n^+) = \underline{dch}(m^+,n^+) = \begin{cases} \mathbb{N} & m=n \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Def A $\underline{\text{directed chain complex}}$ is

$$\begin{tikzcd} & K_2^+ \arrow[d, hook] & K_1^+ \arrow[d, hook] & K_0^+ \arrow[d, hook] & \\ \cdots & K_2 \arrow[r] & K_1 \arrow[r] & K_0 \arrow[r] & 0 \end{tikzcd}$$

chain complex.

Remark

$$\underline{DCh_{\mathbb{Z}}} \overset{f.f.}{\hookrightarrow} \underline{cMonCat}(\underline{dch}, \underline{cMon})$$

@ Directed chain complexes produce ∞-categories

$$\mu : \underline{DCh} \longrightarrow \underline{\infty\text{-}Cat}^{str}$$

$$(K, K^+) \longmapsto \mu(K, K^+)$$

n-cells : $x = \begin{pmatrix} x & s_{n-1}x & \cdots & s_0 x \\ x & t_{n-1}x & \cdots & t_0 x \end{pmatrix}$ s.t. $t_k x - s_k x$

$$\underset{K_n^+}{\rotatebox{90}{$\in$}} \qquad \underset{K_{n-1}^+}{\rotatebox{90}{$\in$}} \qquad \underset{K_0^+}{\rotatebox{90}{$\in$}}$$

$$= \partial_k(s_{k+1}x) = \partial_k(t_{k+1}x)$$

$$(s_n x = t_n x := x)$$

k-th source / target :

$$s_k x = \begin{pmatrix} s_k x & s_{k-1}x & \cdots & s_0 x \\ s_k x & t_{k-1}x & \cdots & t_0 x \end{pmatrix}$$

$$t_k x = \begin{pmatrix} t_k x & s_{k-1}x & \cdots & s_0 x \\ t_k x & t_{k-1}x & \cdots & t_0 x \end{pmatrix}$$

k-th composition:

$$y *_k x = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccccccc} & n & \cdots & & k & \cdots & \\ & & & & & & \end{array}$$

$$y *_k x = \left(\begin{array}{ccccccc} & n & \cdots & & k & \cdots & \\ x+y & s_{n-1}x + s_{n-1}y & \cdots & s_k x & s_{k-1}x & \cdots & s_0 x \\ x+y & t_{n-1}x + t_{n-1}y & \cdots & t_k y & t_k y & \cdots & t_0 y \end{array}\right)$$

$(s_k y = t_k x)$

Example

- $$\begin{tikzcd} & \mathbb{N} \arrow[d, hook] & \mathbb{N} \arrow[d, hook] & \mathbb{Z}/2\mathbb{Z} \arrow[d, hook, "\shortparallel"] & \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "6"] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \end{tikzcd}$$

- $$\begin{tikzcd} \bar{0} \arrow[r, bend left=40, "m"] \arrow[r, bend right=40, "n"'] \arrow[r, phantom, "\Downarrow q"] & \bar{1} \end{tikzcd}$$

$\bar{m} = \bar{1} - \bar{0} = \bar{1}$

$\bar{n} = \bar{1} - \bar{0} = \bar{1}$

$6q = n - m$

- $$\begin{tikzcd} \bar{0} \arrow[r, bend left=60, "l"] \arrow[r, "m" description] \arrow[r, bend right=60, "n"'] \arrow[r, phantom, shift left=3, "\Downarrow p"] \arrow[r, phantom, shift right=3, "\Downarrow q"] & \bar{1} \end{tikzcd} = \begin{tikzcd} \bar{0} \arrow[r, bend left=50, "l"] \arrow[r, bend right=50, "n"'] \arrow[r, phantom, "\Downarrow q+p"] & \bar{1} \end{tikzcd} , \quad \begin{tikzcd} \bar{0} \arrow[r, bend left=40, "s"] \arrow[r, bend right=40, "t"'] \arrow[r, phantom, "\Downarrow r"] & \bar{0} \arrow[r, bend left=40, "m"] \arrow[r, bend right=40, "n"'] \arrow[r, phantom, "\Downarrow q"] & \bar{1} \end{tikzcd} = \begin{tikzcd} \bar{0} \arrow[r, bend left=40, "m+s"] \arrow[r, bend right=40, "n+t"'] \arrow[r, phantom, "\Downarrow q+r"] & \bar{1} \end{tikzcd}$$

@ What are $\mathcal{C} \xrightarrow{\varphi} \mu K$

for $\mathcal{C}$: $\underline{\infty\text{-Cat}}^{str}$, K : $\underline{dCh}$?

$$\begin{tikzcd} \cdots \arrow[r, shift left, "s_n"] \arrow[r, shift right, "t_n"'] & \mathcal{C}_n \arrow[r, shift left, "s_{n-1}"] \arrow[r, shift right, "t_{n-1}"'] \arrow[dd, "\varphi_n"] & \cdots & \mathcal{C}_1 \arrow[r, shift left, "s_0"] \arrow[r, shift right, "t_0"'] \arrow[dd, "\varphi_1"] & \mathcal{C}_0 \arrow[dd, "\varphi_0"] & \\ \cdots & & K_n^+ \arrow[dl] \quad \cdots & & K_1^+ \arrow[dl] & K_0^+ \arrow[dl] \\ \arrow[r, "\partial_n"'] & K_n \arrow[r, "\partial_{n-1}"'] & \cdots & K_1 \arrow[r, "\partial_0"'] & K_0 \arrow[r] & 0 \end{tikzcd}$$

s.t. (1) $\operatorname{Im}(\varphi_n) \subseteq K_n^+ \quad (n \in \mathbb{N})$

(2) $\partial_n \circ \varphi_{n+1} = \varphi_n \circ t_n - \varphi_n \circ s_n$

$$\left(\begin{aligned} \partial_n(\varphi_{n+1}(x)) &= t_n^K \circ \varphi_{n+1}(x) - s_n^K \circ \varphi_{n+1}(x) \\ &= \varphi_n \circ t_n(x) - \varphi_n \circ s_n(x) \end{aligned} \right)$$

Remark

A priori, φ_n looks like

$$\begin{tikzcd} \mathcal{C}_n \arrow[d, "\varphi_n"', "{\langle \varphi_n^n,\ \varphi_n^{n-1},\ \varphi_n^{n-1}, \ldots \rangle}"] \\ K_n \times K_{n-1}^2 \times \cdots \end{tikzcd}$$

,

but φ_n^{n-1} and the rest are determined as

$\varphi_n^{n-1} = \varphi_{n-1} \circ s_{n-1}$

etc.

(3)
$$\begin{tikzcd}
\mathcal{C}_n \times_{\mathcal{C}_m} \mathcal{C}_n \arrow[r, "*_n"] \arrow[d, "\varphi_n \times_{\varphi_m} \varphi_n"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{C}_n \arrow[d, "\varphi_n"] \\
K_n \times_{K_m} K_n \arrow[r, "(-)+(-)"'] & K_n
\end{tikzcd} \qquad (n>m)$$

Lem $\underline{\infty\text{-Cat}}^{str}(\mathcal{C}, \mu -)$ preserves limits. Proof Easy. □

Since $\underline{DCh}$ is presentable, this copresheaf is representable.

$$\underline{\infty\text{-Cat}}^{str}(\mathcal{C}, \mu -) \cong \underline{DCh}(\lambda\mathcal{C}, -)$$

@ Concrete description of $\lambda\mathcal{C}$:

$$(\lambda\mathcal{C})_n := \mathbb{Z}^{\oplus \mathcal{C}_n} \Big/ \overline{x *_m y} = \bar{x} + \bar{y} \qquad (m<n)$$

$$(\lambda\mathcal{C})_{n+1} \xrightarrow{\partial_n} (\lambda\mathcal{C})_n \ ; \quad \bar{x} \longmapsto \overline{t_n x} - \overline{s_n x}$$

$$(\lambda\mathcal{C})^+_n := \langle \bar{x} \mid x \in \mathcal{C}_n \rangle \quad \text{as a comm. mon.}$$

Now, we have $\infty\text{-Cat}^{str} \underset{\mu}{\overset{\lambda}{\underset{\leftarrow}{\overset{\rightarrow}{\perp}}}} dCh$.

Example. $(\lambda *)_0 = \mathbb{Z}, \ (\lambda *)_n = 0 \ (\because \ \bar{1} = \bar{1} + \bar{1})$

$(\lambda *)^+_0 = \mathbb{N}$

Lem $\mathcal{C} \underset{G}{\overset{F}{\underset{\leftarrow}{\overset{\rightarrow}{\perp}}}} \mathcal{D}$ with $\mathcal{C}$: fin. complete. $\Rightarrow$ $\mathcal{C} \underset{R}{\overset{F/F1}{\underset{\leftarrow}{\overset{\rightarrow}{\perp}}}} \mathcal{D}/F1$

☺

$$\begin{tikzcd}
FX \arrow[rr, "f"] \arrow[dr, "F!"'] & \circlearrowright & Y \arrow[dl, "y"] & \Big\| & X \arrow[r, "\tilde{f}"] \arrow[dr, "!"'] & Ry \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & GY \arrow[d, "Gy"] \\
& F1 & & & & 1 \arrow[r, "\eta_1"'] & GF1
\end{tikzcd}$$

□

$$\underline{\infty\text{-}\mathrm{Cat}}^{str} \underset{\nu}{\overset{\lambda}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \underline{DCh}/\lambda* = \underline{ADC}$$

Explicitly,

$$\begin{tikzcd}
 & & & & & K_0^+ \arrow[dl, bend right] & \\
\arrow[r] & K_{n+1} \arrow[r, "\partial_n"] \arrow[d] & K_n \arrow[r] \arrow[d] & \cdots & \arrow[r] & K_0 \arrow[d, "\varepsilon_0"'] \arrow[rr] \arrow[dr, phantom, "\circlearrowright"] & {} \arrow[u] \arrow[d] & 0 \\
 & & & & & & \mathbb{N} \arrow[dl, bend left] & \\
\arrow[r] & \cancel{0} \arrow[r] & \cancel{0} \arrow[r] & \cdots & & \mathbb{Z} \arrow[rr] & & 0
\end{tikzcd}$$

Remark In the original paper (on arXiv) and the paper by Ara et.al., ADC does not require $\varepsilon_0(K_0^+) \subseteq \mathbb{N}$.

But, this paper (Gepner & Heine does)

$$\begin{tikzcd}
 & & & & & & \mathbb{N} \arrow[dl, bend right] & \\
\lambda* \arrow[dd, hook, "m"'] & =(\cdots \arrow[r] & 0 \arrow[r] \arrow[dd, hook] & \cdots \arrow[r] \arrow[dd, phantom, "\cdots"] & 0 \arrow[r] \arrow[dd, hook] & \mathbb{Z} \arrow[rr] \arrow[dd, "id"'] & {} \arrow[u] \arrow[d] & 0) \\
 & & & & & & \mathbb{Z} \arrow[dl] & \\
\mathbb{Z}[0] & =(\cdots \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[rr] & & 0)
\end{tikzcd}$$

is monic,

so $\underline{ADC} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \underline{ADC}'$ ← their definitions.

For $\mathcal{C} : \infty\text{-}\mathrm{Cat}^{str}$, $\varepsilon_0 : (\lambda\mathcal{C})_0 \to \mathbb{Z}$; $\bar{x} \mapsto 1$

@ Concrete description of ν

$$\begin{tikzcd}
\nu(K,\varepsilon) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mu K \arrow[d] \\
* \arrow[r, "1"'] & \mu\lambda*
\end{tikzcd}$$

in $\infty\text{-}\mathrm{Cat}^{str}$

$$\mu\lambda* = \{0 \overset{\circlearrowright^{id}}{,} 1 \overset{\circlearrowright^{id}}{,} 2 \overset{\circlearrowright^{id}}{,} \cdots\}$$

$\nu(K,\varepsilon)$ is the subcat. of μK consisting of

$$\begin{pmatrix} x & s_{n-1}x & \cdots & s_0 x \\ x & t_{n-1}x & \cdots & t_0 x \end{pmatrix} \quad \text{s.t. } \varepsilon(s_0 x) = \varepsilon(t_0 x) = 1 .$$

@ Bases on ADCs

Def A <u>basis</u> of an $(K, K^+) \in \underline{DCh}$ is a family of sets $(B_n)_{n \in \mathbb{N}}$ s.t. $K_n^+ = \mathbb{N}^{\oplus B_n}$, $K_n = \mathbb{Z}^{\oplus B_n}$.

When (K, K^+) has a basis, every $c \in K_n$ is uniquely written as $c = c^+ - c^-$ where $c^+, c^- \in K_n^+$

In particular, we write $\partial_n^+ c := (\partial_n c)^+$, $\partial_n^- c := (\partial_n c)^-$.

$$\begin{tikzcd} & K_n^+ \arrow[d, hook] \\ K_{n+1} \arrow[ur, shift left, "\partial_n^+"] \arrow[ur, shift right, "\partial_n^-"'] \arrow[r, "\partial_n"'] & K_n \end{tikzcd} \quad ; \quad \partial_n = \partial_n^+ - \partial_n^-$$

We define $\langle c \rangle_m^+$ & $\langle c \rangle_m^-$ for $c \in K_n$ $(m \leq n)$ as

$$\begin{tikzcd} K_n \arrow[r, shift left, "\partial^+"] \arrow[r, shift right, "\partial^-"'] & K_{n-1} \arrow[r, shift left, "\partial^+"] \arrow[r, shift right, "\partial^-"'] & K_{n-2} \arrow[r, shift left] \arrow[r, shift right] & \cdots \end{tikzcd}$$

n-th source & target !

$$c := \langle c \rangle_n^+ \overset{\partial^+}{\longmapsto} \langle c \rangle_{n-1}^+ \overset{\partial^+}{\longmapsto} \langle c \rangle_{n-2}^+$$

$$c := \langle c \rangle_n^- \underset{\partial^-}{\longmapsto} \langle c \rangle_{n-1}^- \underset{\partial^-}{\longmapsto} \langle c \rangle_{n-2}^-$$

Def For $(K, K^+, \varepsilon) \in \underline{ADC}$, a basis on (K, K^+) is <u>unital</u> if

$$K_0^+ \xrightarrow{\varepsilon} \mathbb{N}$$

$$\langle c \rangle_0^+ \mapsto 1, \quad \langle c \rangle_0^- \mapsto 1$$

Remark

- $\partial^+(-c) - \partial^-(-c) = \partial(-c) = -\partial c$
  $= \partial^- c - \partial^+ c$
  $\therefore \partial^+ - = -\partial^-, \ \partial^- - = -\partial^+$
- $\partial^2 = (\partial^+ - \partial^-)(\partial^+ - \partial^-)$
  $= (\partial^+)^2 - \partial^-\partial^+ - \partial^+\partial^- + (\partial^-)^2$
  $= (\partial^+)^2 + (\partial^-)^2 - 2\partial^+\partial^- = 0$
  $\therefore \partial^+\partial^- = 1/2((\partial^+)^2 + (\partial^-)^2)$

Rough idea of $\nu(K, K^+, \varepsilon)$ from a unital basis.
(this only works for loop-free case)

<u>0-cells</u> $b \in B_0$

<u>atomic 1-cells</u> $\langle b\rangle_0^- \xrightarrow{b} \langle b\rangle_0^+$ for $b \in B_1$

<u>atomic 2-cells</u>

$$\begin{tikzcd}
\langle b\rangle_0^- \arrow[rr, bend left, dashed, "\langle b\rangle_1^-"] \arrow[rr, bend right, dashed, "\langle b\rangle_1^+"'] & \Downarrow b & \langle b\rangle_0^+
\end{tikzcd}$$

for $b \in B_2$

$\vdots$

◎ Steiner's theorem

Thm

$$\begin{tikzcd}
\underline{\infty\text{-}\mathrm{Cat}}^{str} \arrow[r, shift left, "\lambda"] \arrow[r, phantom, "\perp"] & \underline{\mathrm{ADC}} \arrow[l, shift left, "\nu"] \\
\underline{\infty\text{-}\mathrm{Cat}}^{\mathrm{Steiner}} \arrow[u, hook] \arrow[r, phantom, "\simeq:"] & \underline{\mathrm{ADC}}^{\mathrm{Steiner}} \arrow[u, hook]
\end{tikzcd}$$

is an idempotent adjunction. For this, see nLab.

The fixed points are exactly Steiner categories.

◎ Monoidal structure on $\underline{\mathrm{ADC}}$

$$\underline{\mathrm{ADC}} \xrightarrow{\text{faithful}} \underline{\mathrm{Ch}}(\underline{\mathrm{Ab}}) = \underline{\mathrm{AbCat}}(\underline{\mathrm{ch}}, \underline{\mathrm{Ab}})$$

Remember, in $\underline{\mathrm{Ch}}(\underline{\mathrm{Ab}})$,

$$(K_\bullet \otimes L_\bullet)_i = \bigoplus_{0 \leq k \leq i} (K_k \otimes L_{i-k}).$$

Day convolution!

Similarly, we can define

$$\begin{tikzcd}
((K, K^+) \otimes (L, L^+))_i = \bigoplus_{0 \leq k \leq i} (K_k \otimes L_{i-k}) \\
((K, K^+) \otimes (L, L^+))_i^+ = \bigoplus_{0 \leq k \leq i} \mathrm{Im}(K_k^+ \otimes L_{i-k}^+) \arrow[u, hook]
\end{tikzcd}$$

Remark

① $$\begin{tikzcd} K_k^+ \arrow[d, tail] \\ K_k \end{tikzcd} \otimes \begin{tikzcd} L_{i-k}^+ \arrow[d, tail] \\ L_{i-k} \end{tikzcd} = \begin{tikzcd} K_k^+ \otimes L_{i-k}^+ \arrow[d, "\text{not inj.!}"] \\ K_k \otimes L_{i-k} \end{tikzcd}$$

Abelianization + image-fact.

② $$\begin{tikzcd} \underline{dCh} \arrow[r, shift left, hook] \arrow[r, phantom, "\perp"] & \underline{cMonCat}(\underline{dch}, \underline{cMon}) \arrow[l, shift left] \end{tikzcd}$$

Day involution theorem tells you the above construction is correct!

Since $\lambda * \in \underline{Ch}$ has a monoid structure $(1, \times)$

$\underline{ADC} = \underline{DCh}/\lambda*$ has a monoidal structure induced by it.

Def $\boxtimes$ is the mon. str. on $\underline{\infty\text{-Cat}}^{\text{Steiner}}$ transported from $\underline{ADC}$.

Example $\square^1 = \begin{tikzcd} \bullet \arrow[d] \\ \bullet \end{tikzcd}$, $\square^2 = \begin{tikzcd} \bullet \arrow[d] \\ \bullet \end{tikzcd} \boxtimes \begin{tikzcd} \bullet \arrow[d] \\ \bullet \end{tikzcd}$, $\square^n = (\square^1)^{\boxtimes n}$

$\wr$

$$\begin{tikzcd} & \mathbb{N} \arrow[d, hook] & & \mathbb{N}^2 \arrow[d, hook] \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "\begin{pmatrix} 1 \\ -1 \end{pmatrix}"] \arrow[d, equal] & & \mathbb{Z}^2 \arrow[d, "+"] \\ & \mathbb{Z}^3/\mathbb{Z}^2 & & \mathbb{Z} \arrow[r] & 0 \end{tikzcd} \otimes \begin{tikzcd} & \mathbb{N} \arrow[d, hook] & & \mathbb{N}^2 \arrow[d, hook] \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "\begin{pmatrix} 1 \\ -1 \end{pmatrix}"] \arrow[d, equal] & & \mathbb{Z}^2 \arrow[r] \arrow[d, "+"] & 0 \\ & \mathbb{Z}^3/\mathbb{Z}^2 & & \mathbb{Z} \arrow[r] & 0 \end{tikzcd} = \begin{tikzcd} & \mathbb{N} \arrow[d, hook] & \mathbb{N}^4 \arrow[d, hook] & \mathbb{N}^4 \arrow[d, hook] \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}^4 \arrow[r] & \mathbb{Z}^4 \arrow[r] \arrow[d, "+"'] & 0 \\ & & & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

@ Join

In $\underline{ADC}$,
$$\begin{tikzcd} A \otimes \lambda(\partial \mathbb{D}^1) \otimes B \arrow[r, "\langle id \otimes \varepsilon \otimes \varepsilon, \varepsilon \otimes \varepsilon \otimes id \rangle"] \arrow[d] & A \oplus B \arrow[d] \\ A \otimes \lambda(\mathbb{D}^1) \otimes B \arrow[r] & A \diamond B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

Accordingly, we have $*$ on $\underline{\infty\text{-Cat}}^{\text{Steiner}}$.

Example $\Delta^n := \mathbb{D}^0 * \cdots * \mathbb{D}^0$: oriental.

$\wr$

$$\begin{tikzcd} & & & & \bigoplus_{[k] \hookrightarrow [n]} \mathbb{Z} \arrow[d, hook] & & & \bigoplus_{[0] \hookrightarrow [n]} \mathbb{N} \arrow[d, hook] \\ \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & \bigoplus_{[k] \hookrightarrow [n]} \mathbb{Z} \arrow[r] & \cdots \arrow[r] & & \bigoplus_{[0] \hookrightarrow [n]} \mathbb{Z} \arrow[d, "+"] \\ & & & & & & & \mathbb{Z} \end{tikzcd}$$

Why augmentation? We want to understand $\mathcal{C} \in \infty\text{-Cat}^{str}$ that are "nicely generated by small data (=basis)".

$$\{\text{atomic } n\text{-cells}\} \xleftrightarrow{1:1} \{b \in K_n \mid b : \text{basic}\}$$

However, 0-cells are all basic by nature and there is no composition of them, so we need to make sure that all 0-cells are single base elmts, not their sums.

This difference b/w $\begin{array}{c}\text{single 0-elements}\\ \shortparallel\\ \text{0-cells}\end{array}$ vs $\begin{array}{c}\text{sums of 0-elements}\\ \shortparallel\\ \text{multiple 0-cells}\end{array}$ is reflected in $\eta_* : * \longrightarrow \mu\lambda *$ of the adjunction $\underline{\infty\text{-Cat}}^{str} \underset{\mu}{\overset{\lambda}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \underline{\mathrm{DCh}}$.

For $\mathcal{C} \in \underline{\infty\text{-Cat}}^{str}$, the condition that $\mathcal{C}$ is perfectly recoverable as the subcategory of $\mu(\lambda\mathcal{C})$ spanned by single basic 0-elements amounts to saying that

$$\begin{tikzcd}
\mathcal{C} \arrow[rd, "\cong"] \arrow[rrd, bend left, "\eta_{\mathcal{C}}"] \arrow[rdd, bend right, "!"'] & & \\
& \bullet \arrow[r] \arrow[d] \arrow[rd, phantom, "\lrcorner", very near start] & \mu\lambda\mathcal{C} \arrow[d, "\mu\lambda !"] \\
& 1 \arrow[r, "\eta_1"'] & \mu\lambda 1
\end{tikzcd}$$

.

This $\searrow$ is the component at $\mathcal{C}$ of the unit of

$$\begin{tikzcd}
& \underline{\mathrm{ADC}} \arrow[d, equal] \\
\underline{\infty\text{-Cat}}^{str} \arrow[r, shift left=1.5, "\lambda"] \arrow[r, phantom, "\perp"] & \underline{\mathrm{DCh}}/\lambda * \arrow[l, shift left=1.5, "\nu"]
\end{tikzcd}$$

.

Some keywords I omitted this time : bases of Steiner categories,
loop-free complexes.
unital