“Sheaves on (deformations of) Θ”

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   really

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Motivation

The Plan

Think of (∞-)cats as presheaves on a dense family of oriented polytopes satisfying a gluing condition that encodes composition.

Warmup

What is a category?

$\mathcal{C}$ is a small cat.

$$
* \quad \left(\cdots\cdots\ \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \;\substack{\longrightarrow \\ \xrightarrow{c} \\ \longrightarrow}\; \mathcal{C}_1 \;\substack{\xrightarrow{s} \\ \xleftarrow{e} \\ \xrightarrow{t}}\; \mathcal{C}_0 \right)
$$

in $\underline{\mathrm{Set}}$

such that some diagrams commute...

$$
[n] \longmapsto (0 \to 1 \to \cdots \to n)
$$

$$
* \quad
\begin{tikzcd}
\Delta \arrow[r, "\sigma"] \arrow[d, hook] & \underline{\mathrm{CAT}} \arrow[dl, bend left] \\
\hat{\Delta} \arrow[ur, dashed, bend left, "R"] \arrow[r, phantom, "\perp"] &
\end{tikzcd}
$$

$N(\mathcal{C}) = \mathrm{Fun}(\sigma(-), \mathcal{C})$ is fully faithful

(Δ is dense in $\underline{\mathrm{Cat}}$)

* Every small cat is a colimit of $\underset{\sim}{\sigma([n])}$'s ...

$\hookrightarrow \quad \varepsilon : R \circ N \Rightarrow 1$ is a iso,

$$R(N(\mathcal{C})) \cong \mathcal{C}$$

$$\cong$$

$$\operatorname{colim}_{El(N(\mathcal{C}))} (\sigma \circ \pi)$$

$$\begin{array}{ccc} El(N(\mathcal{C})) & & \\ \downarrow \pi & & \\ \Delta & \longrightarrow & CAT \end{array}$$

<u>Localization</u>:

$$\underline{Cat} \underset{N}{\overset{R}{\rightleftarrows}} \Delta \quad \cdots$$

$R(X) = Ho(X)$ is the homotopy cat of a sSet, X.

<u>Image of the Nerve</u>:

If $\mathcal{C}$ is a cat.

$\{ \rightarrow \ \rightarrow \ \ldots \ \rightarrow \}$

$$\begin{array}{ccc} \Delta_1 \amalg_{x_0} \Delta_1 \ldots \amalg_{x_0} \Delta_1 & \longrightarrow & N(\mathcal{C}) \\ i_n \downarrow & \nearrow_{\exists !} & \\ \Delta_n & & \end{array}$$

spine inclusions.

(because composition in $\mathcal{C}$ is strictly assoc.)

<u>THM</u>: $X \in \hat{\Delta}$ is the nerve of a cat iff

$$\underbrace{X_n \xrightarrow{\cong} X_1 \times_{X_0} \cdots \times_{X_0} X_1}_{\text{Induced by spine inclusion(s).}}$$

i.e. Hom$(i, N(C))$ is a iso.

First Main Theorem(s)

A Geometric Presentation for (anti)Oriented Cats: [1, Theorem 1.10.1]

The nerve functor

$$\begin{tikzcd} \boxtimes\mathrm{Cat} \arrow[r] \arrow[r, phantom, shift left=2, "\perp"] & \mathrm{Fun}\left(\Theta(\square, \boxtimes)^{\mathsf{op}}, \infty\mathrm{Gpd}\right) \arrow[l, bend right, dashed] \end{tikzcd}$$

is a fully faithful right adjoint and presheaves in its image $\underbrace{\text{satisfy two Segal conditions...}}_{\text{"glory"}}$

First Main Theorem(s)

A Nerve Theorem: [1, Theorem 1.10.3]

The oriented nerve functor

$$\begin{tikzcd} \infty\mathrm{Cat} \arrow[r] \arrow[r, phantom, shift left=3, "\perp"] & \mathrm{Fun}\left(\mathcal{O}^{\mathsf{op}}, \infty\mathrm{Gpd}\right) \arrow[l, bend right, "L"'] \end{tikzcd}$$

is a fully faithful right adjoint and its image consists of presheaves that are *local* with respect to

- boundary decomposition
- top-cell decomposition
- globular decomposition

Sheaves on a Site

What is a site?

$\mathcal{C}$ a cat.

A <u>sieve</u> is a subfunctor $S \hookrightarrow \text{よ}_c = \mathcal{C}(-,c)$.

A Grothe. Topology, J, is a collection of sieves we call "J-covering" satisfying some conditions ....

- maximal sieves are covering:
- pb stable:

$$\begin{tikzcd} f^*R \arrow[r, "J"'] \arrow[d, hook] & R \arrow[d, hook] \\ \text{よ}_C \arrow[r, "f_*"'] & \text{よ}_D \end{tikzcd}$$

- transitivity ...

Examples

Examples of Topologies

- (open cover) $\mathcal{C} = \mathrm{Open}(X)$, J_{open} generated by open covers:

$$S \in J_{open}(U) \iff S \text{ contains an open cover: } \mathcal{U} = \{U_i \hookrightarrow U \mid i \in I\}$$

$$\begin{tikzcd} & \mathcal{C}/c \arrow[d, phantom, "\cong" rotate=90] \\ \mathrm{El}\, S \arrow[ur, hook] \arrow[r, hook] \arrow[dr, "\pi"'] & \mathrm{El}\, X_c \arrow[d, "\pi"] \\ & \mathcal{C} \end{tikzcd} \qquad \Bigg| \qquad \begin{tikzcd} \mathcal{C}^{op} \arrow[r, bend left, "S"] \arrow[r, bend right, "X_c"'] \arrow[r, phantom, "\Downarrow"] & \underline{\mathrm{Set}} \end{tikzcd}$$

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$$S \hookrightarrow \mathcal{X}_u$$

$$\{x \hookrightarrow u\}$$

- (Dense/double-neg) $\mathcal{C}$ a category, $J_{\neg\neg}$ defined by

$$S \in J_{\neg\neg}(C) \iff \forall\, f : B \to C, \exists\, g : A \to B, \text{ such that } f \circ g \in S$$

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- (Atomic) $\mathcal{C}$ a category, $J_{@}$ defined by

$$S \in J_{@}(C) \iff S \neq \varnothing$$

What are sheaves?

$(\mathcal{C}, J)$ a site, $F : \mathcal{C}^{op} \longrightarrow \underline{Set}$

Def: A matching family is a transformation $S \xrightarrow{\alpha} F$

$$\left\{\begin{tikzcd} x \arrow[r] \arrow[d] & y \arrow[dl] \\ c & \end{tikzcd}\right\} \longmapsto \left\{\begin{array}{c} \alpha_x \in F(x) \\ \uparrow \\ \alpha_y \in F(y) \end{array}\right\}$$

Def: A presheaf, F, is a sheaf if every matching family extends uniquely (for all $S \in J(c)$)

$$\begin{tikzcd} S \arrow[r, "\alpha"] \arrow[d, hook] & F \\ よ_c \arrow[ur, dotted, "\exists !"'] & \end{tikzcd}$$

$$\left\{\begin{tikzcd} & u \cap v \arrow[dl] \arrow[dr] & \\ u \arrow[dr] & & v \arrow[dl] \\ & c & \end{tikzcd}\right. \quad (c = u \cup v) \quad \rightsquigarrow \quad \begin{tikzcd} & \alpha_{u \cap v} & \\ \alpha_u \arrow[ur, mapsto] & & \alpha_v \arrow[ul, mapsto] \\ & \alpha \in F(c) \arrow[ul, mapsto] \arrow[ur, mapsto] \arrow[uu, mapsto] & \end{tikzcd}$$

Examples

Example

Let $\mathcal{C} = \mathrm{Open}(X)$ and define $C^\alpha(U) = \{f : U \to \mathbb{R} \mid f$ is α times differentiable.$\}$.
This is a sheaf when X is a C^α manifold for $0 \leq \alpha \leq \infty$.

Examples

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(example)

Example

Let $\mathcal{C}$ be a category, the canonical topology is the largest topology for which all representables are sheaves.

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Example

Let $\mathcal{C}$ be a category, the canonical topology is the largest topology for which all representables are sheaves.

Example (VI - [2])

Sheaves for the dense topology are used in logic to construct models of set theory that break stuff

The associated sheaf functor

Theorem (III.5.1 - [2])

$(\mathcal{C}, J)$ a site,

$$\begin{tikzcd} Sh(\mathcal{C}, J) \arrow[r, hook] \arrow[r, phantom, "\perp", shift left=2] & \hat{\mathcal{C}} \arrow[l, bend right, dashed, "(-)^{++}"'] \end{tikzcd}$$

is fully faithful and has a left-exact left adjoint.

Idea: See [MLiM III.5.1] for proof.

$$\begin{tikzcd} \hat{\mathcal{C}} \arrow[r, "(-)^+"] \arrow[rr, bend left, "(-)^{++}"] & \hat{\mathcal{C}}_{sep} \arrow[r, "(-)^+"] & Sh(\mathcal{C}, J) \end{tikzcd}$$

and $(-)^+$ is given by taking "filtered colimits of matching families"

$P : \mathcal{C}^{op} \longrightarrow \mathrm{Set}$,

$$P^{+}(C) = \operatorname*{colim}_{S \in J(C)} \underbrace{\mathrm{Nat}(S, P)}_{}$$

Matching fams $S \Rightarrow P$

(dgm)

$J(C) \hookrightarrow \underline{\underline{\mathrm{Sub}(\mathcal{Y}_C)}}$

poset

Localizations in General

Localizations

Let W be a class of morphisms in a cateory $\mathcal{C}$.

We can always construct the localization $\mathcal{C}[W^{-1}]$...

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "L"'] & & \mathcal{D} \\
& \mathcal{C}[W^{-1}] \arrow[ur, dotted, "\exists!"'] &
\end{tikzcd}$$

← inverts W

$$\mathrm{Fun}_W(\mathcal{C},\mathcal{D}) \cong \mathrm{Fun}(\mathcal{C}[W^{-1}],\mathcal{D}).$$

$$\mathcal{C}[W^{-1}]_0 = \mathcal{C}_0$$

$\mathcal{C}[W^{-1}]_1$ = equivalence classes of zig-zags ← $\sim$ means in W

$$\{x \xleftarrow{\sim} \cdot \to \cdot \xleftarrow{\sim} \cdot \to \cdots \xleftarrow{\sim} \cdot \to y\}$$

Local Objects and Saturated Classes ; $W \subseteq \mathcal{C}$,

$\mathcal{L}_X(f)$

<u>Local objects</u> : X is W-local iff $\mathrm{Hom}(f, X)$ is an iso for all $f \in W$.

<u>Local morphisms</u> : $f : A \longrightarrow B$ in $\mathcal{C}$ is W-local iff $\mathrm{Hom}(f, X)$ is an iso for all W-local objects X.

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[r, "\forall"] & X \\ B \arrow[ur, "\exists !"'] & \end{tikzcd}$$

$$\mathrm{Hom}(B, X) \xrightarrow[f^*]{\cong} \mathrm{Hom}(A, X)$$

$W \longrightarrow$ (J-local Morphisms) $\longrightarrow$ (Maps Inverted by L)

If "=" say "W is saturated"

<u>THM</u> : "=" iff $\mathcal{C} \xrightarrow{L} \mathcal{C}[W^{-1}]$ has an adjoint

Example: Sheaves

Let $(\mathcal{C}, J)$ be a site. , J is a collection of subfunctors $S \hookrightarrow よ_C$

<u>J-local objects</u>:

$$\begin{tikzcd} S \arrow[r, "\forall"] \arrow[d, hook] & F \\ よ_C \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

i.e. these are the J-sheaves on $\mathcal{C}$.

<u>J-local morphisms</u>: well

$$\begin{tikzcd} \mathrm{Sh}(\mathcal{C}, J) \arrow[r, hook, bend right] \arrow[r, phantom, "\perp"] & \hat{\mathcal{C}} \arrow[l, dashed, bend right, "(-)^{++}"'] \end{tikzcd}$$

, so

J-local morphisms are precisely the maps that get inverted by $(-)^{++}$.

$$\hookrightarrow \quad \frac{F \overset{\eta}{\hookrightarrow} F^{++}}{F^{++} \xrightarrow{\cong} (F^{++})^{++}}$$

Sheaves that are not Sheaves

but they are...

An elementary topos is . . .

a finitely complete and cocomplete cartesian closed category with subobject classifier, Ω.

$\mathrm{Sh}(\mathcal{C},J)$

you have all small colimits, not just finite.

$\mathrm{Sub}(X) \cong \mathrm{Hom}(X,\Omega)$

An elementary topos is . . .

a finitely complete and cocomplete cartesian closed category with subobject classifier, Ω.

A Lawvere-Tierney topology is. . .

a map $j : \Omega \to \Omega$ such that → allows us to talk about things being "locally true"

- $j \circ \text{true} = \text{true}$
- $jj = j$
- $j \circ \wedge = \wedge \circ (j \times j)$

where $\wedge : \Omega \times \Omega$ is the meet of $\mathrm{Sub}(\Omega)$.

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\text{true}"] \arrow[dr, "j"'] \arrow[dr, phantom, shift left=3, "="] & \Omega \arrow[d, "j"] \\ & \Omega \end{tikzcd}$$

"if P is true, then P is locally true"

Examples

Example

Take $\mathcal{C} = \mathrm{Open}(\mathrm{X})$, then $\widehat{\mathcal{C}}$ is an elementary topos

- $\Omega \in \widehat{\mathcal{C}}$ is defined by
$$\Omega(U) = \{S \mid \text{ sieves on } U\}$$
- $J \hookrightarrow \Omega$ is defined by
$$J(U) = \{S \mid \text{ sieves that cover } U\}$$
- $j : \Omega \to \Omega$ is defined by
$$j_U(S) = \{W \subseteq U \mid S \cap W \text{ covers } W\}$$

$j_U(S)$ is picking out the largest open covered by S

Closure Operator

A LT-topology j induces a closure operator on subobject lattices...

$$\mathrm{Sub}(X) \xrightarrow{\ \overline{(-)}\ } \mathrm{Sub}(X)$$

$$\begin{pmatrix} U \\ \downarrow \\ X \end{pmatrix} \rightsquigarrow
\begin{tikzcd}
\overline{U} \arrow[rr] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner", very near start] & & 1 \arrow[d, "\mathrm{true}"] \\
X \arrow[r, "\chi_U"'] & \Omega \arrow[r, "j"'] & \Omega
\end{tikzcd}$$

A subobj. is dense if $U \cong \overline{U}$.

Sheaves revisited

A j-sheaf in $\mathcal{E}$ is a local object for the class of j-dense monomorphisms in $\mathcal{E}$.

Theorem

The category of j-sheaves on $\mathcal{E}$, $Sh_j(\mathcal{E})$ is a topos with

$$Sh_j(\mathcal{E}) \hookrightarrow \mathcal{E}$$

a geometric embedding whose left adjoint is called 'sheafification.'

Theorem

If $\mathcal{E} \hookrightarrow \mathcal{F}$ is a geometric embedding, then there exists a LT-topology on $\mathcal{F}$ such that $\mathcal{E} \simeq \mathrm{Sh}_j(\mathcal{F})$

$$\begin{tikzcd} \mathcal{E} \arrow[r, hook, "\perp", "f.f."'] & \mathcal{F} \arrow[l, dashed, bend right, "lex"'] \end{tikzcd}$$

Sketches

Theories

0-ary op's 1-ary op's. ...

A theory is...

a signature, $\Sigma = (\Omega_0, \Omega_1, \dots)$, of n-ary operations with a set of equations that encode the axioms.

Theories

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A model of a theory . . .

interprets the signature in a category $\mathcal{C}$ such that the diagrams determined by the equations commute.

Theories

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A model of a theory ...

interprets the signature in a category $\mathcal{C}$ such that the diagrams determined by the equations commute.

Lawvere's category of 'clones'

allowed him to *define* theories as categories with products and models as product-preserving functors (into $\mathbf{Set}$).

Sketches

A sketch is a tuple $(\mathcal{G}, \mathcal{D}, \mathcal{L}, \mathcal{C})$ where

- $\mathcal{G}$ is a graph
- $\mathcal{D}$ is a set of diagrams in $\mathcal{G}$

a small cat...

- $\mathcal{L}$ is a set of cones
- $\mathcal{C}$ is a set of cocones.

Models of Sketches

A model of a sketch (in Set) is a graph homomorphism

$$M : \mathcal{G} \to \mathrm{Set}$$

such that, after applying M

- the diagrams of $\mathcal{D}$ commute
- the cones from $\overset{\mathcal{L}}{\cancel{\mathcal{C}}}$ become limits
- the cocones from $\underset{\mathcal{C}}{\cancel{\mathcal{L}}}$ become colimits

$\Bigg\}$: M(cone) is a limit cone, in Set
M(cocone) is a colimit cone

Categories of Models

- A morphism between models is basically a natural transformation.
- Models of a sketch form a category.
- A category is ‘sketchable’ if it is the category of models of a sketch.

Examples

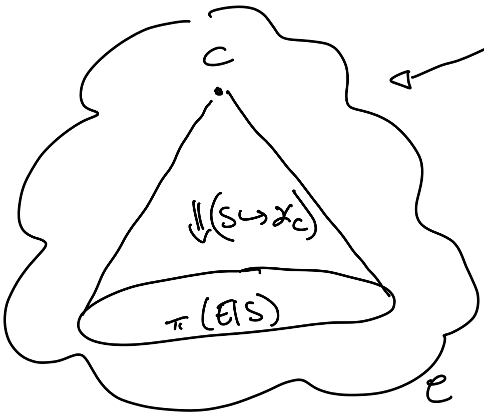
Here are some examples

- models of $(\mathcal{G}, \varnothing, \varnothing, \varnothing)\ldots$
- permutations
- M-sets, for a fixed monoid M
- monoids, groups, abelian groups, rings, etc.
- torsion free abelian groups
- small cats
- sheaves on a site

$\Downarrow$

$(\mathcal{C}, J)$ a $\underset{\curlyvee}{\text{(small)}}$ site.

Define a sketch, $(\mathcal{C}^{op}, \mathcal{L}_J)$



$$\begin{tikzcd} (E|S)^{op} \arrow[r, hook] \arrow[dr, "\pi"'] & (\mathcal{C}/c)^{op} \arrow[d, "\pi"] & \\ & \mathcal{C}^{op} \arrow[r, bend left, "S"] \arrow[r, bend right, "x_c"'] \arrow[r, phantom, "\Downarrow"] & \underline{\mathrm{Set}} \end{tikzcd}$$

Then models are presheaves $\mathcal{C}^{op} \xrightarrow{F} \underline{\mathrm{Set}}$

such that $F(\text{cone}) = (\text{limit cone})$.

i.e. $$F(c) \longrightarrow \prod_{\substack{u \to c \\ \in S}} F(u) \rightrightarrows \prod_{\substack{v \to u \to s \\ \in S}} F(u)$$

is an equalizer. This is another form of the sheaf cond'n.

<u>Cor</u>: $Sh(\mathcal{C}, J)$ is a cat. of models of a limit sketch.

Characterizing Presentable/Accessible (∞)-categories

~~Example~~ THM

Presentable categories are categories of models of limit sketches (Gabriel-Ulmer Duality)

$$(\mathrm{Lex})^{op} \longrightarrow (\mathrm{LFP}) \quad \text{is an equivalence of cats.}$$

$$A \longmapsto \mathrm{Lex}(A, \underline{\mathrm{Set}})$$

cat. of models of A.

$$\mathrm{Hom}(X, \underline{\mathrm{Set}}) \longleftarrow X$$

Stone Duality

Characterizing Presentable/Accessible (∞)-categories

Example

Presentable categories are categories of models of limit sketches (Gabriel-Ulmer Duality)

Example

Accessible categories are categories of models of arbitrary sketches (Lair)

Characterizing Presentable/Accessible (∞)-categories

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Presentable categories are categories of models of limit sketches (Gabriel-Ulmer Duality)

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Accessible categories are categories of models of arbitrary sketches (Lair)

Example

Presentable ∞-categories are ∞-categories of models of limit sketches (in spaces) [3]

Characterizing Presentable/Accessible (∞)-categories

Example

Presentable categories are categories of models of limit sketches (Gabriel-Ulmer Duality)

Example

Accessible categories are categories of models of arbitrary sketches (Lair)

Example

Presentable ∞-categories are ∞-categories of models of limit sketches (in spaces) [3]

Example

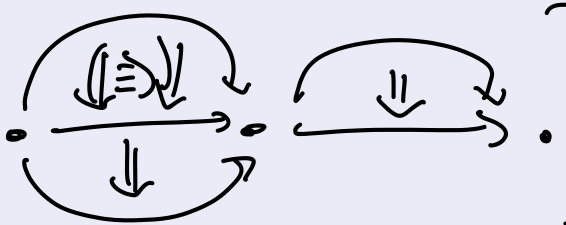
Accessible ∞-categories are ∞-categories of models of arbitrary sketches (in spaces) [3]

Sheaves on Θ

What is Θ?

As a Category, $\Theta \hookrightarrow \infty\mathrm{Cat}$

generated by bipointed V's of Σ's of • (all over ∂D', then forget)

E.g. **Objects**: []

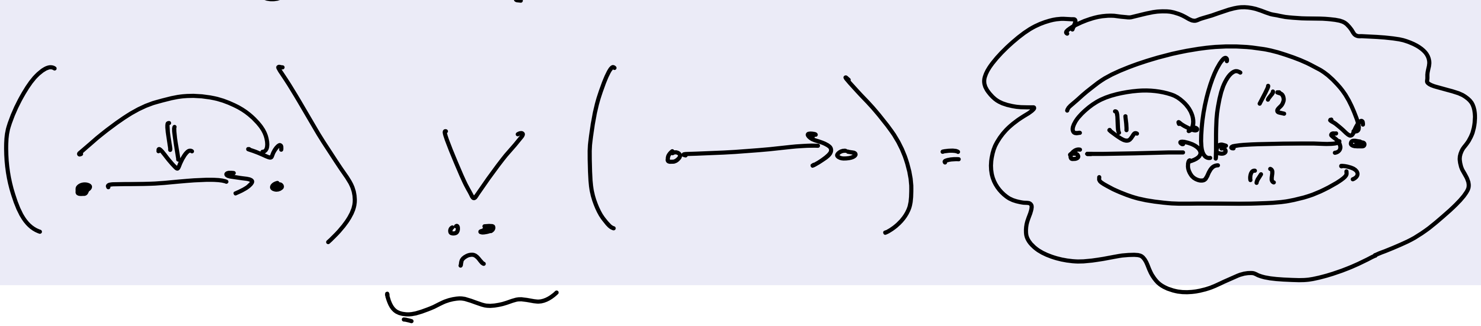
(**Morphisms**: basically functors...)

What is Θ?

As a Construction…

$\mathcal{V}$ is monoidal then you can similarly define

$\Theta(\mathcal{V})$ by "bipointed wedges of suspensions" of obj's in $\mathcal{V}$.



Dense Subcategories of $\infty\mathrm{Cat}$

Def/Thm

$i : \mathcal{C} \hookrightarrow \mathcal{D}$ is dense if the induced nerve functor $\mathcal{D} \xhookrightarrow{よ} \widehat{\mathcal{D}} \xrightarrow{\hat{i}} \widehat{\mathcal{C}}$ is fully faithful.

Dense Subcategories of $\infty\mathrm{Cat}$

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Sufficient Condition for Density - Campion

If $\mathcal{D} \hookrightarrow \infty\mathrm{Cat}$ is closed under retracts and it contains Θ, then $\mathcal{D}$ is dense in $\infty\mathrm{Cat}$.

Dense Subcategories of $\infty\mathrm{Cat}$

Def/Thm

$i : \mathcal{C} \hookrightarrow \mathcal{D}$ is dense if the induced nerve functor $\mathcal{D} \xhookrightarrow{\text{よ}} \widehat{\mathcal{D}} \xrightarrow{\hat{i}} \widehat{\mathcal{C}}$ is fully faithful.

Sufficient Condition for Density - Campion

If $\mathcal{D} \hookrightarrow \infty\mathrm{Cat}$ is closed under retracts and it contains Θ, then $\mathcal{D}$ is dense in $\infty\mathrm{Cat}$.

Lemma

If $\mathcal{C} \hookrightarrow \infty\mathrm{Cat}$ is dense and $\mathcal{C} \hookrightarrow K(\mathcal{D})$, then $\mathcal{D}$ is dense.

$\mathcal{E}$ (For the proof of theorems)

Outline of proof of some main theorems in the paper

Lemma

$\Theta \hookrightarrow \infty\mathrm{Cat}$ is dense

Lemma: Θ's are retracts of oriented simplices are retracts of oriented cubes

There are inclusions into idempotent completions

Retracts of representables:

$$\Theta \hookrightarrow K(\triangle) \qquad \triangle \hookrightarrow K(\Box)$$

dense in ∞Cat.

the left one implies $\triangle$ is dense, the right one then implies $\Box$ is dense.

Strict?

Retract in ∞-cat?

in this sense

Outline of proof of some main theorems in the paper

Why density?

$K(\mathcal{C})$ is the full subcat of retracts of representables so $\mathcal{C}$ embeds fully and faithfully into $K(\mathcal{C})$:

$$\begin{tikzcd} \mathcal{C} \arrow[r, hook] \arrow[rr, bend left, "よ"] & K(\mathcal{C}) \arrow[r, hook] & \widehat{\mathcal{C}} \end{tikzcd}$$

If $\mathcal{D}$ is Cauchy complete (as a Set-enriched cat) then $\mathcal{D}$-valued presheaves on $\mathcal{C}$ coincide with $\mathcal{D}$-valued presheaves on $K(\mathcal{C})$:
`https://ncatlab.org/toddtrimble/published/Karoubi+envelope`

Characterizing the image of the nerve

Theorem 2.8.8 -

If $\mathcal{V} \hookrightarrow \mathcal{W}$ is a small dense full monoidal subcategory, then $\Theta(\mathcal{V}) \hookrightarrow {}_{\mathcal{V}}\mathrm{Cat}$ is dense in ${}_{\mathcal{W}}\mathrm{Cat}$ and the essential image of the $\Theta(\mathcal{V})$-nerve consists of all presheaves that satisfy the Segal condition and all morphism presheaves are in the essential image of the $\mathcal{W}$-nerve

Geometric Presentation of (anti)Oriented Cats

First Main Theorem

A Geometric Presentation for (anti)Oriented Cats: [1, Theorem 1.10.1]

The nerve functors

$\Theta(V)$ for $V = (\square, \boxtimes)$

$$\boxtimes\mathrm{Cat} \to \mathrm{Fun}\left(\Theta(\square, \boxtimes)^{\mathsf{op}}, \infty\mathrm{Gpd}\right)$$

$$\mathrm{Cat}\boxtimes \to \mathrm{Fun}\left(\Theta(\overline{\square}, \overline{\boxtimes})^{\mathsf{op}}, \infty\mathrm{Gpd}\right)$$

$$\infty\mathrm{Cat} \to \mathrm{Fun}\left(\Theta(\square, \times)^{\mathsf{op}}, \infty\mathrm{Gpd}\right)$$

are fully faithful right adjoints and their images consist of the presheaves satisfying two Segal conditions.

- $\boxtimes\mathrm{Cat}$ is the category of 'oriented categories,' i.e. the category of $(\infty\mathrm{Cat}, \boxtimes)$-enriched cateogries
- $\Theta(\square, \boxtimes)^{\mathsf{op}}$ is the Θ construction of $(\square, \boxtimes)$
- For any $(s, t) \in F(*)$, the morphism presheaf from s to t is given by

$$\begin{tikzcd} \square^{op} \arrow[r, "F\circ\Sigma"] \arrow[rr, bend left, "\mathrm{LMor}_F(s{,}t)"] & \infty\mathrm{Gpd}_{/F(*)\times F(*)} \arrow[r, two heads, "\pi"] & \infty\mathrm{Gpd} \end{tikzcd}$$

The Segal Condition

For all $k, n_1, \ldots, n_k \in \mathbb{N}$,

$$F\left(\bigvee_i \Sigma(\Box^{n_i})\right) \xrightarrow{\sim} F(\Sigma(\Box^{n_0})) \times_{F*} F(\Sigma(\Box^{n_1})) \times_{F*} \cdots \times_{F*} F(\Sigma(\Box^{n_k}))$$

is a weak equivalence.

induced by a spine inclusion.

$\underline{\Theta(\vee)}$

$$X_{\alpha_1} \amalg X_{\alpha_2} \amalg \ldots \amalg X_{\alpha_n}$$

$$\downarrow$$

$$X$$

$$\rightsquigarrow$$

The Local Segal Condition

For any $(s, t) \in F(*)$, the presheaf $\mathrm{LMor}_F(s, t)$ is local with respect to

The Local Segal Condition

For any $(s,t) \in F(*)$, the presheaf $\mathrm{LMor}_F(s,t)$ is local with respect to

- the boundary decomposition:

$$\mathrm{coeq}\left(\coprod_{1\leq i<j\leq n} 4 \times N(\square^{n-2}) \rightrightarrows \coprod_{0\leq i\leq n} 2 \times N(\square^{n-1})\right) \to N(\partial\square^n)$$

The Local Segal Condition

For any $(s,t) \in F(*)$, the presheaf $\mathrm{LMor}_F(s,t)$ is local with respect to

- the boundary decomposition:

$$\mathrm{coeq}\left(\coprod_{1\leq i<j\leq n} 4 \times N(\square^{n-2}) \rightrightarrows \coprod_{0\leq i\leq n} 2 \times N(\square^{n-1})\right) \to N(\partial\square^n)$$

- the top cell decomposition:

$$N(\mathbb{D}^n) \coprod_{\partial N(\mathbb{D}^n)} N(\partial\square^n) \to N(\square^n)$$

The Local Segal Condition

- the globular decomposition:

$$\left(N(\mathbb{D}^{i_0}) \coprod_{N(\mathbb{D}^{j_1})} N(\mathbb{D}^{i_1}) \coprod_{N(\mathbb{D}^{j_2})} \cdots \coprod_{N(\mathbb{D}^{j_n})} N(\mathbb{D}^{i_n}) \right) \to N \left(\mathbb{D}^{i_0} \coprod_{\mathbb{D}^{j_1}} \mathbb{D}^{i_1} \coprod_{\mathbb{D}^{j_2}} \cdots \coprod_{\mathbb{D}^{j_n}} \mathbb{D}^{i_n} \right)$$

$$\begin{tikzcd}
0 \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2, "\Downarrow"] \arrow[r, phantom, shift right=2, "\Downarrow"] & 1 \arrow[r] \arrow[r, bend left=60] \arrow[r, phantom, shift left=2, "\Downarrow"] & 2 \arrow[r] & 3
\end{tikzcd} \longleftarrow [3; 2, 1, 0]$$

cocomplete

$$\begin{tikzcd}
\Theta_2 \arrow[r, hook] \arrow[d] & \underline{Str2Cat} \arrow[dl, bend left, "N"] \arrow[dl, phantom, "\perp"] \\
\hat{\Theta}_2 \arrow[ur, dashed, bend left, "R"] &
\end{tikzcd}$$

$N \leftarrow$ is fully faithful?

[Every strict 2-cat. is a colimit of these Θ_2-spaces - Bergner Ozornova Rovelli;]

<u>Image of the Nerve</u> ?

$\hookrightarrow$ $[\Theta_2\text{-spaces} : \Theta_2^{op} \longrightarrow \hat{\Delta}]$ $\leftarrow$ [Explicit comparison between 2-complicial & 2-segal...]

$$\text{hspine}([n ; q]) = \Theta_2[1; q_1] \coprod_{\Theta_2[1,0]} \cdots \coprod_{\Theta_2[1,0]} \Theta_2[1; q_n]$$

$$\text{hspine}(3;2,1,0) \cong \left[\begin{tikzcd}
0 \arrow[r, bend left=60] \arrow[r] \arrow[r, bend right=60] \arrow[r, phantom, shift left=2, "\Downarrow"] \arrow[r, phantom, shift right=2, "\Downarrow"] & 1 \arrow[r] \arrow[r, bend left=60] \arrow[r, phantom, shift left=2, "\Downarrow"] & 2 \arrow[r] & 3
\end{tikzcd}\right] \hookrightarrow \Theta_2[3;2,1,0].$$

$$\mathrm{vspine}([1;k]) = \bigcup \mathrm{spine}(\Delta[k])$$

local objects wrt spines are ~~Segal~~ Segal.

$$\begin{tikzcd} \mathrm{spine} \arrow[d, hook] \\ \mathcal{L}_c \end{tikzcd}$$

$$\Delta \qquad (\Delta \wr \Delta) \qquad (\Delta \wr \Delta \wr \Delta)$$

$$\begin{tikzcd} \Theta_1 \arrow[r, hook, "i"'] & \Theta_2 \arrow[l, bend right] \arrow[r, hook, "i"'] & \Theta_3 \arrow[l, bend right] \arrow[r, hook, "i"'] & \cdots \end{tikzcd} \qquad \mathrm{colim}(i) = \Theta$$

$$\begin{tikzcd} \mathrm{Psh}(\Theta_1) \arrow[r, bend left] & \mathrm{Psh}\,\Theta_2 \arrow[l, "i^*"] \arrow[r, bend left] & \mathrm{Psh}\,\Theta_3 \arrow[l, "i^*"] \arrow[r, bend left] & \cdots \arrow[l, "i^*"] \end{tikzcd} \qquad \lim(i^*) = \mathrm{Psh}\,\Theta$$

$$\mathrm{Fc}^L(\mathrm{colim}\,\Theta_n^{op}, \mathcal{G}pd) \cong \lim \mathrm{Fc}^L(\Theta_n^{op}, \mathcal{G}pd)$$

Thanks

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