Spaces, $\mathcal{S}$ will replace Set. $\mathcal{S}$ is cartesian closed, so can enrich. $Cat_0 := \mathcal{S}$, $Cat_{n+1} := Cat_n$-enriched categories

Recall: $Cat_n \simeq (\infty, n)Cat$.

$$Cat_\infty := \underset{n\to\infty}{\operatorname{colim}^{Pr^L}} \left(\begin{tikzcd} \mathcal{S} \arrow[r, hook] & Cat_1 \arrow[r, hook] & \cdots \arrow[r, hook] & Cat_n \arrow[r, hook] & Cat_{n+1} \arrow[l, bend right, "\tau"', "\perp"] \arrow[l, bend left, "\iota", "\perp"'] \arrow[r, hook] & \cdots \end{tikzcd} \right)$$

This is a (1-) category whose objects are given by sequences $(C_0, C_1, \ldots)$ w/ $C_i \in Cat_i$ and $C_n \simeq \iota C_{n+1}$.

- Want: "oriented versions" of homotopy theory.

E.g. $\mathcal{S}_*$ = pointed spaces. S^1 the circle & $\forall X \in \mathcal{S}$,

$$\Omega X := Maps_{\mathcal{S}_*}(S^1, X)$$

Alternatively: homotopy pullback
$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner h"] & \bullet \arrow[d] \\ \bullet \arrow[r] \arrow[ur, phantom, "\cong"] & X \end{tikzcd}$$

or strict pullback
$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & PX \arrow[d] \\ \bullet \arrow[r] & X \end{tikzcd}$$

Write $S^1 = B\mathbb{Z}$. Can we "directify" by replacing $\mathbb{Z}$ w/ $\mathbb{N}$?

I.e. for $\mathcal{C} \in Cat_{\infty_*}$, should we define $\Omega\mathcal{C}$ as $Maps_{Cat_{\infty_*}}(B\mathbb{N}, \mathcal{C})$.

These give endo's at base pts like we want, but Cat_{∞_*} is an $(\infty,1)$ category, so $Maps_{Cat_{\infty_*}}(B\mathbb{N}, \mathcal{C}) \in \mathcal{S} \rightsquigarrow$ only has invertible higher cells, so not really "$\mathcal{C}(pt, pt)$".

$$\mathrm{Maps}_{\mathrm{Cat}_\infty}(\mathcal{C},\mathcal{D}) = \begin{cases} \text{obj} = \text{functors } \mathcal{C}\to\mathcal{D} \\ \text{mor} = \text{pseudonat'l transformation} \\ \text{etc.} \end{cases}$$

Solution; replace pseudo w/ (op)lax:

$\Omega_b\mathcal{C} := \mathrm{Maps}^{lax}_{\mathrm{Cat}_\infty}(B\mathbb{N}, \mathcal{C})$, so we'll need some kind of oriented / (op)lax pullback;

$$\begin{tikzcd} \Omega_b\mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \;\; \Nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \mathcal{C} \end{tikzcd}$$

In non-pointed setting,

$$\begin{tikzcd} \mathcal{C}(x,y) \arrow[r] \arrow[d, "x"'] \arrow[dr, phantom, "\ulcorner \;\; \Nearrow"] & \bullet \arrow[d, "y"] \\ \bullet \arrow[r] & \mathcal{C} \end{tikzcd}$$

"comma pullback"

$$\begin{tikzcd} \{Fx \to Gy\} \arrow[r] \arrow[d, "F"'] \arrow[dr, phantom, "\ulcorner \;\; \Nearrow"] & \mathcal{C} \arrow[d, "G"] \\ \mathcal{D} \arrow[r] & \mathcal{E} \end{tikzcd}$$

Prop[1]: $\mathrm{Maps}^{lax}_{\mathrm{Cat}_\infty}(\mathcal{C},\mathcal{D})$ exists; i.e. there's a monoidal structure $\boxtimes$ on Cat_∞ and it has an adjoint:

$$\mathrm{Cat}_\infty(\mathcal{C}\boxtimes\mathcal{D}, \mathcal{E}) \simeq \mathrm{Cat}_\infty(\mathcal{C}, \mathrm{Cat}_\infty^{lax}(\mathcal{D},\mathcal{E}))$$

Cat_∞ has $(\geq)\,2$ closed monoidal structures, $\times$ & $\boxtimes$, thus it is self-enriched in $(\geq)\,2$ ways. $\Omega_b : \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ is not a functor of $(\mathrm{Cat}_\infty, \times)$-categories, but is a functor of $(\mathrm{Cat}_\infty, \boxtimes)$-categories. $\boxtimes : \mathrm{Cat}_\infty \times \mathrm{Cat}_\infty \to \mathrm{Cat}_\infty$ is called the **Gray product**. $(\mathrm{Cat}_\infty, \boxtimes)$-categories are called **oriented categories**. $\exists\, \Sigma \dashv \Omega_b$ w/ $\Sigma = B\,\mathrm{Free}_\otimes(-)$

More on $\boxtimes$:

$\square^0 = *$

$\square^1 = \bullet \longrightarrow \bullet$

$$\square^2 = \begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$$

etc.

$\square^1 \boxtimes \square^1$ defined by: $\mathrm{Cat}_\infty(\square^1 \boxtimes \square^1, \mathcal{C})$

$$\simeq \mathrm{Cat}_\infty(\square^1, \mathrm{Cat}_\infty^{\mathrm{lax}}(\square^1, \mathcal{C}))$$

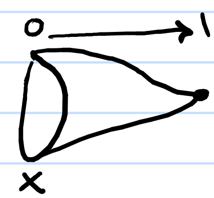
Since $\square^1 = \bullet \longrightarrow \bullet$ & $\mathrm{Cat}_\infty^{\mathrm{lax}}(\square^1, \mathcal{C})$ has objects morphisms in $\mathcal{C}$ & morphisms lax transformations. Thus $\square^1 \boxtimes \square^1$ picks out squares in $\mathcal{C}$ of the form $\begin{tikzcd} \cdot \arrow[r, "\eta"] \arrow[d, "F"'] \arrow[dr, phantom, "\nearrow"] & \cdot \arrow[d, "G"] \\ \cdot \arrow[r, "\eta"'] & \cdot \end{tikzcd}$, so $\square^1 \boxtimes \square^1 = \square^2$.

More generally, $\square^n \boxtimes \square^m = \square^{n+m}$

Joins

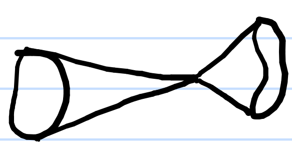
For spaces; start w/ cone:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & X \times [0,1] \arrow[d] \\ \bullet \arrow[r] & CX \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd} \rightsquigarrow$$



Join is two cones glued at tips:

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & CX \arrow[d] \\ CY \arrow[r] & X * Y \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$



For categories; lax cone

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\cong"] & X \boxtimes \square^1 \arrow[d] \\ \bullet \arrow[r] & CX \arrow[ul, phantom, "\lrcorner", very near start] \end{tikzcd}$$

$$\begin{tikzcd} \Omega C \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \ \Nearrow", very near start] & \bullet \arrow[d] \\ \bullet \arrow[r] & C \end{tikzcd} \quad \underline{\text{or}} \quad \begin{tikzcd} \Omega C \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \ \text{strict!}", very near start] & \mathrm{Cat}_\infty^{\mathrm{lax}}(\square^1, C) \arrow[d] \\ \bullet \arrow[r] & C \end{tikzcd}$$

Similarly

$$\begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow \ \lrcorner", very near end] & \bullet \arrow[d] \\ \bullet \arrow[r] & \Sigma C \end{tikzcd} \quad \underline{\text{or}} \quad \begin{tikzcd} C \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{strict!} \ \lrcorner"] & \bullet \arrow[d] \\ C \boxtimes \square^1 \arrow[r] & \Sigma C \end{tikzcd}$$

Remark: $\boxtimes$ is <u>not</u> symmetric. $\boxtimes^{\mathrm{rev}}$ has inner hom given by $\mathrm{Cat}_\infty^{\mathrm{oplax}}(-,-)$

$\rightsquigarrow$ Bi-enrichment; enrich over $\boxtimes$ & $\boxtimes^{\mathrm{rev}}$ at same time.