Univalent Categories

There's a commuting diagram

$$\begin{tikzcd}
 & \mathrm{Cat}^{\mathrm{flagged}} \arrow[dl, shift right] \arrow[dr, shift left] & \\
\mathrm{Cat}^{\mathrm{strict}} \arrow[ur, shift right] \arrow[dr, shift right] & & \mathrm{Cat}^{\mathrm{univalent}} \arrow[ul, shift left] \arrow[dl, shift left] \\
 & \mathrm{Cat}^{\mathrm{gaunt}} \arrow[ul, shift right] \arrow[ur, shift left] &
\end{tikzcd}$$

{Categories w/ sets (or any class) of objects} $= \mathrm{Cat}^{\mathrm{strict}}$

We'll want a functor $\mathrm{Set} \xrightarrow{\mathrm{Disc}} \mathrm{Cat}$ sending $X \mapsto C_0 = X$
& $C_1 = \coprod_{x \in X} \{id_x\}$

$C \in \mathrm{Cat}^{\mathrm{strict}}$ includes data of $\mathrm{Ob}(C) \in \mathrm{Set}$ and an ESO functor

$$cl(C): \mathrm{Disc}(\mathrm{Ob}(C)) \longrightarrow C$$

$C, D \in \mathrm{Cat}^{\mathrm{strict}}$, then a morphism is a functor $F: C \longrightarrow D$
w/ function $\mathrm{Ob}(F): \mathrm{Ob}(C) \to \mathrm{Ob}(D)$ and 2-cell

$$\begin{tikzcd}
\mathrm{Disc}(\mathrm{Ob}(C)) \arrow[r] \arrow[d, "\mathrm{Disc}(\mathrm{Ob}(F))"'] \arrow[dr, phantom, "cl(F)\ \Leftarrow"] & C \arrow[d, "F"] \\
\mathrm{Disc}(\mathrm{Ob}(D)) \arrow[r] & D
\end{tikzcd}$$

Strict functors $(F, \mathrm{Ob}(F), cl(F))$ & $(G, ob(G), cl(G))$ are equal if
$ob(F) = ob(G)$ and $\exists$ nat'l iso $F \cong G$ s.t.

$$\begin{tikzcd}
\mathrm{Disc}(\mathrm{Ob}(C)) \arrow[r] \arrow[d, "\mathrm{Disc}(ob(G))"'] \arrow[dr, phantom, "cl(G)\ \Leftarrow"] & C \arrow[d, "G"] \\
\mathrm{Disc}(\mathrm{Ob}(D)) \arrow[r] & D
\end{tikzcd}
=
\begin{tikzcd}
\mathrm{Disc}(\mathrm{Ob}(C)) \arrow[r] \arrow[d, "\mathrm{Disc}(ob(G))"', "="] \arrow[dr, phantom, "cl(F)\ \Leftarrow"] & C \arrow[d, "F"', "\cong \gamma"] \arrow[d, bend left=60, "G"] \\
\mathrm{Disc}(\mathrm{Ob}(D)) \arrow[r] & D
\end{tikzcd}$$

The 1-category $\mathrm{Cat}^{\mathrm{strict}}$ is itself a strict category. A nat'l transformation $\eta: (F, ob(F), cl(F)) \longrightarrow (G, ob(G), cl(G))$ is just $\eta: F \Rightarrow G$

Categories (in the usual sense) & functors form a 1-cat.

—— || —— + nat'l transformations is a (2,2)-cat.

$\exists$ functor (of 2-cats) $\{\text{cat} + \text{functors}\} \longrightarrow \{\text{cat} + \text{functors} + \text{nat'l transf}\}$

but it is certainly not ESO on 2-cells.

Can we factor this thru an equiv of 2-cat's ?

$$\begin{tikzcd}
\{C_0 + C_1\} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \{C_0 + C_1 + C_2\} \\
\text{other 2-cat} \arrow[ur, "\text{surjective on } C_0, C_2, C_3"']
\end{tikzcd}$$

Something about this new functor is a "fibrant replacement" of the old functor.

This other 2-cat is $\mathrm{Cat}_{\text{strict}}$

- Univalence: "$X \cong Y \iff X = Y$". A category C is univalent if $\forall$ functor $F : \mathrm{Iso} \longrightarrow C$,

$$\begin{tikzcd}
\mathrm{Iso} \arrow[r, "!"] \arrow[d, "F"'] \arrow[dr, phantom, "="] & * \arrow[dl, dashed, "\exists G"] \\
C
\end{tikzcd}$$

Note that this is not invariant under equivalences $C \simeq D$.

If C is a strict category, then the above diagram implies that if $A \cong B \in C$, then $A = B \in \mathrm{ob}(C)$ and $\cong$ is $=$.

↪ something about $\simeq$ in $\mathrm{Cat}_{\text{strict}} \overset{?}{\longleftrightarrow} \cong$ in $\mathrm{Cat}_{\text{usual}}$

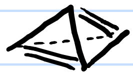
- X a set, and $x, y \in X$. $\mathrm{Id}(x,y) = \emptyset$ or $\{=\}$. Abstractly, we use the diagonal map $X \to X \times X$;

$$\begin{tikzcd}
\mathrm{Id}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & * \arrow[d] \\
X \arrow[r, "\Delta"] & X \times X
\end{tikzcd}$$

In spaces, either compute Π_n or do a fibrant replacement $X \longmapsto$ paths in X

So
$$\begin{tikzcd} \mathrm{Id}(x,y) \arrow[r, "!"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & * \arrow[d, "{(x,y)}"] \\ \mathrm{maps}([0,1],X) \arrow[r, "{(s,t)}"'] & X\times X \end{tikzcd}$$
defines the space of paths from x to y in X.

For $\mathcal{C}\in \mathrm{Cat}_{(\infty,\infty)}$, $\mathcal{C}_0 \in \mathrm{Cat}_{(\infty,0)} = \text{spaces}$, then putting $\mathcal{C}_0$ in place of X gives the space of identifications from x to y.

Can similarly define the space of isomorphisms x to y via
$$\begin{tikzcd} \mathrm{Iso}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner h"] & * \arrow[d, "{(x,y)}"] \\ \mathcal{C}_{1,\mathrm{Iso}} \arrow[r, "{(s,t)}"'] & \mathcal{C}_0\times\mathcal{C}_0 \end{tikzcd}$$
In simplicial models, $\mathcal{C}_{1,\mathrm{iso}}$ are the maps  $\longrightarrow \mathcal{C}$

In general, $\mathrm{Id}(x,y)\hookrightarrow \mathrm{Iso}(x,y)$ b/c there's a map of diagrams
$$\begin{tikzcd} \mathcal{C}_0 \arrow[r, "\Delta"] \arrow[d, hook] & \mathcal{C}_0\times\mathcal{C}_0 \arrow[d, equal] & * \arrow[l] \arrow[d, equal] \\ \mathcal{C}_{1,\mathrm{iso}} \arrow[r, "{(s,t)}"'] & \mathcal{C}_0\times\mathcal{C}_0 & * \arrow[l] \end{tikzcd}$$
. $\mathcal{C}\in\mathrm{Cat}_\infty$ is univalent when the inclusion $\mathrm{Id}(x,y)\hookrightarrow\mathrm{Iso}(x,y)$ is an iso (in spaces, so a weak homotopy equiv) $\forall x,y\in\mathcal{C}_0$.

Theo's puzzles

- $\mathcal{C}\in\mathrm{Cat}_{(\infty,1)}$. $\mathrm{Set} = \mathrm{Spaces}_{\leq 0}$ so $\mathrm{Set}\hookrightarrow\mathrm{Spaces}$ is f.f.
  ↳ quasi-cat's

$X\in\mathcal{C}$ is called 0-truncated if
$$\begin{tikzcd} & \mathrm{Set} \arrow[d, hook, shift right] \\ \mathcal{C}^{op} \arrow[ur, dashed, "\exists"] \arrow[r, "{\mathcal{C}(-,x)}"'] & \mathrm{Set} \arrow[u, shift right, "\pi_0"'] \arrow[u, phantom, "\dashv"] \end{tikzcd}$$
If $\exists$ ⇢, then it is unique up to homotopy.

e.g. $S\in\mathrm{Cat}_{(\infty,0)}$ is 0-truncated iff it is homotopy equivalent to a set.

If $\mathcal{C}$ is presentable, $\mathcal{C}_{\leq 0} \underset{\perp}{\rightleftarrows} \mathcal{C}$. In particular,
$$\begin{tikzcd} (\mathrm{Cat}_{(\infty,1)})_{\leq 0} \arrow[r, hook, shift right] \arrow[d, phantom, "\cong"] & \mathrm{Cat}_{(\infty,1)} \arrow[l, shift right] \arrow[l, phantom, "\perp"] \arrow[r, hook, shift right] & \mathrm{Cat}^{flag}_{(\infty,1)} \arrow[l, shift right] \arrow[l, phantom, "\perp"] \arrow[r, shift left] & (\mathrm{Cat}^{flag}_{(\infty,1)})_{\leq 0} \arrow[l, hook, shift left] \arrow[l, phantom, "\perp"] \arrow[d, phantom, "\cong"] \\ \mathrm{Cat}_1^{gaunt} \arrow[rrr, hook, shift right] & & & \mathrm{Cat}_1^{strict} \arrow[lll, shift right] \arrow[lll, phantom, "\perp"] \end{tikzcd}$$

$$\begin{tikzcd}[row sep=large] \mathrm{Cat}^{\mathrm{flag}}_\infty \arrow[d, shift right=1.5ex] \arrow[d, phantom, "\dashv"] & = \text{functors } \Theta \longrightarrow \text{Spaces w/ Segal axiom} \\ \mathrm{Cat}_\infty \arrow[u, hook, shift right=1.5ex] & = \text{functors } \Theta \longrightarrow \text{Spaces w/ Segal \& Rezk axioms.} \end{tikzcd}$$

Questions:

1. Who is $(\mathrm{Cat}^{\mathrm{flag}}_\infty)_{\leq 0}$? Is it $\mathrm{Cat}^{\mathrm{strict}}_\infty$?

2. Is ??? ESO ?