Univalence vs Rezk completeness

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Let $\mathcal{S}$ be the category of spaces[1] and let X be a category object in $\mathcal{S}$. I picture that like this:

$$
\begin{array}{c}
X_{\text{eq}} \\
\downarrow \\
X_0 \rightrightarrows X_1 \longleftarrow X_1 \times_{X_0} X_1.
\end{array}
$$

Here $X_{\text{eq}} \subset X_1$ is the subspace of invertible morphisms, but I won't define that now. The only place where it comes into place is in Proposition 4 below. In fact, from now on every map and diagram is in $\mathcal{S}$. This is because we are going to work *within* X (even though we could phrase many maps below as functors into X).

Example 1. The terminal category is $*$. $\triangle$

An *object* in X is a map $x : * \to X_0$ in $\mathcal{S}$.

Definition 2. The **identity type** between objects x and y is the space defined by the following pullback in $\mathcal{S}$.:

$$
\begin{array}{ccc}
\text{Id}(x,y) & \longrightarrow & * \\
\downarrow & \lrcorner & \downarrow{\scriptstyle y} \\
* & \xrightarrow{x} & X_0
\end{array}
$$

Definition 3. The **type of equivalences** between objects x and y is the space defined by the following pullback in $\mathcal{S}$.:

$$
\begin{array}{ccc}
\text{Eq}(x,y) & \longrightarrow & * \\
\downarrow & \lrcorner & \downarrow{\scriptstyle (x,y)} \\
X_{\text{eq}} & \xrightarrow{(s,t)} & X_0 \times X_0
\end{array}
$$

Proposition 4. *The type of equivalences is equivalent to the following pullback:*

$$
\begin{array}{ccc}
* \times_{X_{\text{eq}}} * & \longrightarrow & * \\
\downarrow & \lrcorner & \downarrow{\scriptstyle \text{id}_y} \\
* & \xrightarrow{\text{id}_x} & X_{\text{eq}}
\end{array}
$$

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[1]Category means $(\infty, 1)$-category.

Proof. `sorry` (you need to use the definition of X_{eq} here) $\qquad\qquad\qquad\qquad\square$

The identity map $\mathrm{id} : X_0 \to X_{\mathrm{eq}}$ induces a map $\mathrm{id}_{x,y} : \mathrm{Id}(x, y) \to \mathrm{Eq}(x, y)$.

Definition 5 (HoTT univalence). A category is **HoTT univalent** if $\Phi_{x,y} : \mathrm{Id}(x, y) \to \mathrm{Eq}(x, y)$ is an equivalence for any $x, y \in X$. [2]

Definition 6 (Rezk completeness). A category is **Rezk complete** if $\mathrm{id} : X_0 \to X_{\mathrm{eq}}$ is an equivalence.

Notation 7. Let X be a space. We use the following notation for the space of paths $x \rightsquigarrow y$:

$$
\begin{array}{ccc}
X(x,y) & \longrightarrow & * \\
\downarrow & \quad\lrcorner & \downarrow y \\
* & \xrightarrow{\ x\ } & X
\end{array}\ .
$$

A map $f : X \to Y$ in $\mathcal{S}$ induces a map $f_{x,y} : X(x, y) \to Y(fx, fy)$.

Lemma 8. *The following are equivalent for a map $f : X \to Y$ in $\mathcal{S}$.*

- *f is an equivalence.*

- *$\pi_0(f)$ is surjective and $f_{x,y}$ is an equivalence for all $x, y \in X$.*

Proof. ($\implies$) Equivalences in $\mathcal{S}$ are stable under pullback. ($\impliedby$) We use the Whitehead theorem: $f_{x,x} : \Omega(X, x) =: X(x, x) \xrightarrow{\sim} Y(y, y) := \Omega(Y, y)$ shows that f is an isomorphism on $\pi_{\geq 1}$, and $\varnothing \neq Y(fx, fy) \implies \varnothing \neq X(x, y)$ shows that f is injective on π_0. $\qquad\square$

Corollary 9. *Rezk completeness implies HoTT univalence.*

Proof. HoTT univalence is the condition that $\mathrm{id} : X_0 \to X_{\mathrm{eq}}$ induces equivalences on path spaces. $\qquad\square$

Remark 10. The additional request that $\pi_0(\mathrm{id})$ is a surjective on π_0 is the assertion that X_{eq} does not have "extra equivalences". Otherwise it could be the case that HoTT holds in that $\mathrm{Id}(x, y) \xrightarrow{\sim} \mathrm{Eq}(x, y)$, but X_{eq} has random connected components which are not seen by id.

Definition 11. (tentative) A **non-univalent category** (aka valent category, aka flagged category) is a category which is not Rezk complete. Note that the *functor* $X_0 \to X$ is essentially surjective.

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[2]See Axiom 2.10.3 in the HoTT book.