Oriented Category Theory Reading Group: Univalent Categories

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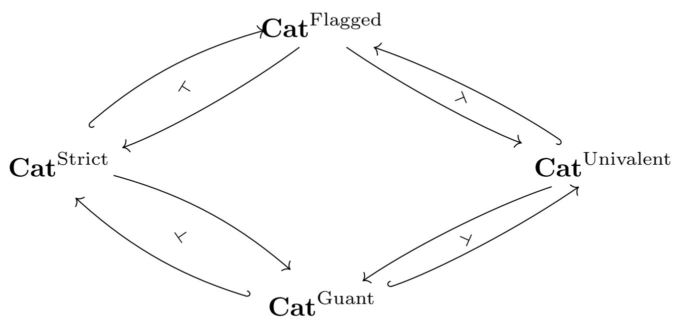
1 Introduction

In standard set-theoretic mathematics, we often take advantage of the fact that the objects in our category form a set (or at least a proper class). When working in non-standard foundations, such as homotopy type theory (HoTT), it becomes less clear what it means to have a *set* of objects. This question can be answered through the theory of *strict* categories. In standard set-theoretic mathematics, we often embrace certain fictions as well, such as the practice of treating isomorphisms as equalities. In non-standard foundations, such as HoTT, it requires some care to define precisely what this fiction is telling us. For example, we must first identify the ways in which two objects x and y can be identical, and the ways in which two objects x and y can be isomorphic. Formally, we will need to define a type $\mathrm{Id}(x, y)$ and a type $\mathrm{Iso}(x, y)$, which can be thought of as *proofs of identification* and *proofs of equivalence*. This leads to a more topological notion of *identity*, and also gives rise to an inclusion $\mathrm{Id}(x, y) \to \mathrm{Iso}(x, y)$. The fiction that isomorphisms are equalities is to pretend that this inclusion is in fact an isomorphism. When this fiction is in fact a reality, we say that a category is *univalent*.

Unsurprisingly, strictness and univalence are independent concepts. That is to say that a category can be weak and univalent, or strict and valent. When a category is both strict and univalent, we say that the category is guant. Given a category with sufficient information (e.g., a flagged category), it is possible to both strictify the category and to Rezk complete [1] the category (i.e., complete the category with respect to univalence). Moreover, strictification and Rezk completion commute. This gives rise to the following diagram, in which the square of inclusions commutes, and the square of completions also commutes.



The goal of these notes is to understand this diagram.

TODO: I should say a bit more about flagged categories (see [2]).

TODO: I occasionally used nlab as a source, since they did not provide references and I could not find any other sources. It would be nice to figure out at some point whether other sources exist, and if the nlab page agrees with the existing sources.

2 Strict Categories

When we first learn undergraduate-level mathematics, we often take for granted that each category will have an underlying set of objects. Eventually, we will learn about Russel's paradox and discovert that this is a logical contradiction. For example, the collection of objects in the category of sets forms a proper class, as opposed to a set. Of course, up to size issues, we can still pretend that every category has a *set* of objects. At some point, every mathematician will learn that set theory is only one of many possible foundations to mathematics. From here onward, it is no longer a trivial statement that every category has an underlying set (resp. proper class) of objects. This leads to the notion of *strict categories*, *strict functors*, and *strict natural transformations*.

We will assume that we have some notion of collections with some way to construct elements within the collection. If we construct an element x of X, then we will write $x : X$, which is often read x *is of type* X. We will not assume a notion of equality for a collection, unless this is stated explicitly.

Definition 2.1. A **category** $\mathcal{C}$ consists of the following data.

- A collection $\mathcal{C}_0$ of *objects*.
- For each pair of objects x and y which inhabit $\mathcal{C}_0$, a collection of *arrows* denoted $\mathcal{C}(x, y)$ with a notion of equality.
- For each pair of arrows $f : \mathcal{C}(x, y)$ and $g : \mathcal{C}(y, z)$, an arrow $g \circ f : \mathcal{C}(x, z)$.

- For each object x, an arrow $1_x : \mathcal{C}(x, x)$.

This data is subject to the following axioms.

- If $f : \mathcal{C}(x, y)$, $g : \mathcal{C}(y, z)$, and $h : \mathcal{C}(z, w)$, then $h \circ (g \circ f) = (h \circ g) \circ f$.
- If $f : \mathcal{C}(x, y)$, then $1_y \circ f = f = f \circ 1_x$.

A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of the following data.

- For each $x : \mathcal{C}_0$ an assignment of $F.x : \mathcal{D}_0$.
- For each $f : \mathcal{C}(x, y)$, an assignment of $F.f : \mathcal{D}(F.x, F.y)$

This data is subject to the following axioms.

- If $x : \mathcal{C}_0$, then $F.1_x = 1_{F.x}$.
- If $f : \mathcal{C}(x, y)$ and $g : \mathcal{C}(y, z)$, then $F.(g \circ f) = (F.g) \circ (F.f)$.

A **natural transformation** $\eta : F \Rightarrow G$ consists of the following data.

- For each $x : \mathcal{C}_0$, an assignment of $\eta_X : \mathcal{D}(F.x, F.y)$.

This data is subject to the following axiom.

- If $f : \mathcal{C}(x, y)$, then $G.f \circ \eta_X = \eta_Y \circ F.f$.

In the case of higher-category theory, we will be interested in categories whose collections are h-types. However, the following discussion makes sense with any choice of foundations (e.g., presets or the ZF set theory without choice). We say that a category is *strict* if its collection of objects admits the structure of a set (or proper class). We would like to have an abstract way to describe such a category, without saying explicitly that $\mathcal{C}_0$ is a set. Given a set $X : \mathbf{Set}_0$, it must be the case that the discrete category generated by X has a collection of objects admitting the structure of X. Let $\mathrm{Disc} : \mathbf{Set} \to \mathbf{Cat}$ be the functor which sends each object $x : \mathrm{Disc}(X)_0 = X$ to $f(x) : \mathrm{Disc}(Y)_0 = Y$. If $\mathcal{C}$ has a collection of objects exhibiting the structure of $X \in \mathbf{Set}$, then there should exists a functor $F : \mathrm{Disc}(X) \to \mathcal{C}$ such that F is essentiality surjective. Note that this functor is only essentially surjective, since $\mathcal{C}_0$ need not be a set.

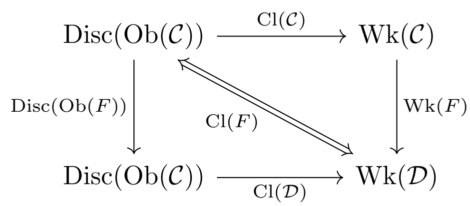
Definition 2.2 ([9])**.** A **strict category** $\mathcal{C}$ consists of the following data.

- A set of objects $\mathrm{Ob}(\mathcal{C}) : \mathbf{Set}_0$.
- A category $\mathrm{Wk}(\mathcal{C}) : \mathbf{Cat}_0$.
- An essentially surjective functor $\mathrm{Cl}(\mathcal{C}) : \mathrm{Disc}(\mathrm{Ob}(\mathcal{C})) \to \mathcal{C}$.

A **strict functor** $F : \mathcal{C} \to \mathcal{D}$ consists of the following data.

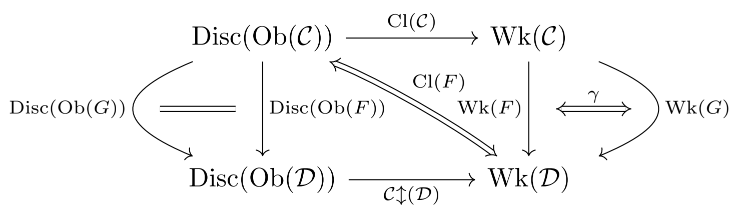
- A function $\mathrm{Ob}(F) : \mathbf{Set}(F.x, F.y)$.
- A functor $\mathrm{Wk}(F) : \mathcal{C} \to \mathcal{D}$.

- A natural isomorphism of the following form.



We say that $F = G$ if $\mathrm{Ob}(F) = \mathrm{Ob}(G)$ and there exists a natural isomorphism $\gamma_X \mathrm{Wk}(F) \cong \mathrm{Wk}(G)$ modulo which $\mathrm{Cl}(F) = \mathrm{Cl}(G)$. A *strict natural transformation* is simply a natural transformation between strict functors.

The equality of strict functors can be best understood through the following concatenation of commuting diagrams.



The concatenation is intended to be a factorization of $\mathrm{Cl}(G)$ in terms of $\mathrm{Cl}(F)$ and γ. A strict natural transformation is the same as a natural transformation, since natural transformations do not rely on defining assignments between collections of object.

Theorem 2.3 ([9])**.** *There is a strict* 2*-category* $\mathbf{Cat}^{\mathbf{Strict}}$ *of strict categories, strict functors, and strict natural transformations.*

TODO: Include an example showing why this is a reasonable/necessary. Also show some examples with h-types

TODO: It seems that our use of the term category does not align perfectly with the use of the word category in homotopy type theory. For example, my definition of a category is (close) to what they would call a precategory. The HoTT book claims that not every strict category is a category, which implies that they place extra conditions on what it means to be a category. This misalignment in terminology may make it hard to transfer results, and may lead to other inconsistencies.

2.1 Univalent Categories

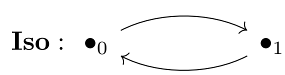
An important concept in type theory and higher category theory is that of the univalence axiom. Intuitively, this axiom states that if $A \cong B$ then $A = B$. On one hand, this axiom may seem absurd, since isomorphism of objects rarely implies equality of objects. On the other hand, this axiom may seem reasonable, since much of mathematics works on the intuition that isomorphic objects are *the same*. That is to say, univalence may seem at first to be a very mysterious axiom. It may help to look at some explanations for why we would want the univalence axiom.

Motivation. In a set-theoretic foundations for mathematics, every mathematical object is assumed to be a set. In other words, every mathematical object has the *type* of a set. In type theory, we instead assume some *universe* of types, in which sets may be but another type [10]. Typically, when people refer to type theory, they are referring to Martin-Löf type theory. There are two flavors of type theory, namely extensional and intensional. These models differ in how we think of equality. For x and y of type A, we can always construct a type $\mathrm{Id}_A(x, y)$ which depends on both x and y. A term $p : \mathrm{Id}_A(x, y)$ can be thought of as a proof that x is identical to y. In the extensional model of type theory, we have the *identity reflexivity rule*, which allows us to derive that if $p : \mathrm{Id}_A(x, y)$ and $b : B(x)$, then $b : B(y)$ [12]. Moreover, models of extensional Martin-Löf type theory are known to be locally Cartesian model categories [11], which are familiar to most mathematicians. Unfortunately, the extensional version of Martin-Löf type theory is known to have an undecidable type-checking problem [4]. If the identity reflexivity rule is discarded, then the intensional version of Martin-Löf type theory is obtained. In the intensional model, we do not assume that if $p : \mathrm{Id}_A(x, y)$ then $x = y$. It was shown by Hofmann and Streicher that there exists a groupoid model of intensional Martin-Löf type theory in which the identity type may contain distinct elements which are not propositionally equal [3]. It was later shown by Warren that models of intensional Martin-Löf type theory correspond to certain $(\infty, 1)$-categories, and that the failure of the identity reflexivity rule is unavoidable [13]. However, certain models of intensional Martin-Löf type theory take this extended notion of identity to its fullest, so that the notion of isomorphism corresponds with the notion of identity. Such categories are said to be univalent.

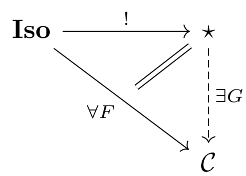
TODO: I would like to make sense of the claim that $p \in \mathrm{Id}(x, y)$ and $b : B(x)$ does not imply that $b : B(y)$. This presumably says something transport. It would be nice to unpack the type-theoretic notions in terms of categories and homotopy theory.

An Important Insight. ([10]) "In particular, one may say that *equivalent types are identical*. However, this phrase is somewhat misleading, since it may sound like a sort of *skeletality* condition which collapses the notion of equivalence to coincide with identity, whereas in fact univalence is about expanding the notion of identity so as to coincide with the (unchanged) notion of equivalence."

As with most mathematical concepts, there are many ways we could define what it means for a category $\mathcal{C}$ to be univalent. The most obvious way is to require that every isomorphism in $\mathcal{C}$ factors through an identity in $\mathcal{C}$. To pick out an isomorphism in $\mathcal{C}$, we can map the walking isomorphism into $\mathcal{C}$.



We then say that $\mathcal{C}$ is univalent if diagrams of the following form always factor through the point (see [5]).

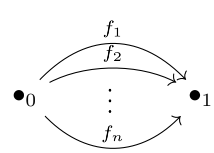


This already allows us to characterize univalent categories within a set theoretic foundation of mathematics.

Theorem 2.4 ([7])**.** *If $\mathcal{C}$ is strict and univalent (i.e.,* gaunt*), then $\mathcal{C}$ is a skeletal category with a thin core.*

Proof. Since $\mathcal{C}$ is strict, then we can work with $\mathcal{C}$ as if it was an ordinary category. First, it must be shown that $\mathcal{C}$ is skeletal. Assume that $x \cong y$. Then there exists an isomorphism $X \xrightarrow{f} Y \xrightarrow{g} X$ between X and Y. Then there exists a functor $F : \text{Iso} \to \mathcal{C}$ such that $F(\bullet_0) = X$ and $F(\bullet_1) = Y$. Since $\mathcal{C}$ is univalent, then this functor factors through $\mathcal{C}$. Then there exists a functor $G : \star \to \mathcal{C}$ such that $F(\bullet_1) = G(\star)$ and $F(\bullet_2) = G(\star)$. This means that $X = F(\bullet_1) = F(\bullet_2) = Y$. Since X and Y were arbitrary, then $X \cong Y$ implies that $X = Y$. It remains to be shown that $\mathcal{C}_0$ has a thin core. Since $\mathcal{C}$ is skeletal, then the only isomorphisms in the core of $\mathcal{C}$ are automorphisms. Assume that $X \xrightarrow{f} X$ is an automorphism of X. Then there exists a functor : $\text{Iso} \to \mathcal{C}$ such that $F(\bullet_0) = F(\bullet_1) = X$, and $F(\bullet_0 \to \bullet_1) = f$. Since $\mathcal{C}$ is univalent, then this functor factors through $\mathcal{C}$. Then there exists a functor $G : \star \to \mathcal{C}$ such that $f = F(\bullet_0 \to \bullet_1) = G(1_\star) = 1_X$. Since f was arbitrary, then $f : X \to X$ an automorphism implies that $f = 1_X$. $\square$

Example 2.5. The follow categories are guant.

- Free categories on graphs.
- Free strict monoidal categories on monoidal signatures.
- Posetal categories.
- The family of small categories: 

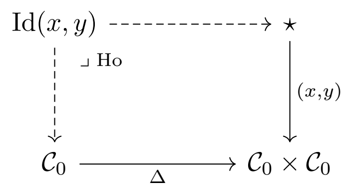
The following categories are not guant.

- Preorder categories which are strictly not posetal.
- The walking isomorphism.

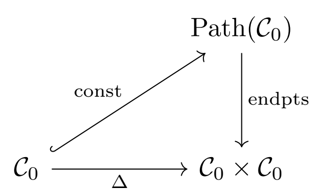
Example 2.6. TODO: nlab claims that Archimedean ordered fields are also guant. Working through this example might be insightful.

It is clear that the map $\mathbf{Iso} \xrightarrow{!} \star \to \mathcal{C}$ picks out an identity $a = a$ and the map $\mathbf{Iso} \to \mathcal{C}$ picks out an isomorphism $x \cong y$. By asking all functors of the form $\mathbf{Iso} \to \mathcal{C}$ to factor through $\star$, we are asking that every identity $a = a$ is equivalent to some isomorphism $x \cong y$. That is to say, $\mathrm{Id}(x, y) \cong \mathrm{Iso}(x, y)$. To make this precise, we must first formally define $\mathrm{Id}(x, y)$ and $\mathrm{Iso}(x, y)$.

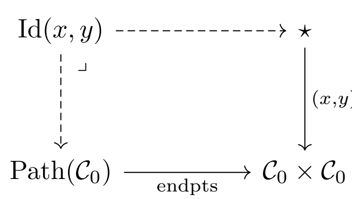
Definition 2.7 ([8])**.** Let $\mathcal{C}$ be a category. Then $\mathrm{Id}(x, y)$ is defined by the following homotopy pullback square.



Intuitively, this pullback picks out objects $a \in \mathcal{C}_0$ such that $a \approx x$ and $a \approx y$. Of course, $\mathcal{C}_0 \xrightarrow{\Delta} \mathcal{C}_0 \times \mathcal{C}_0$ is not a fibration. However, $\mathcal{C}_0 \xrightarrow{\Delta} \mathcal{C}_0 \times \mathcal{C}_0$ factors as follows where $\mathrm{Path}(\mathcal{C}_0)$ is the path space of $\mathcal{C}_0$ and $\mathrm{Path}(\mathcal{C}_0) \xrightarrow{\text{endpts}} \mathcal{C}_0 \times \mathcal{C}_0$ is the fibration which picks out the start and end point of each path through $\mathcal{C}_0$.



Then it suffices to compute the following pullback square instead.



This tells us that the elements of $\mathrm{Id}(x, y)$ are precisely the paths $\gamma : [0, 1] \to \mathcal{C}_0$ through $\mathcal{C}_0$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

To define $\mathrm{Iso}(x, y)$, we must first identify the subspace of all isomorphisms in $\mathcal{C}$. Formally, this is the subspace $\mathcal{C}_{\text{iso}}$ of $\mathcal{C}_1$ defined by picking out the invertible morphisms in $\mathcal{C}_1$. This can be constructed in various ways. Regardless of the construction, it will be true that the identity data $\mathcal{C}_0 \xrightarrow{\text{id}} \mathcal{C}_1$ factors through $\mathcal{C}_{\text{iso}}$.

$$\begin{array}{ccc} & & \mathcal{C}_{\text{iso}} \\ & \overset{\text{id}}{\nearrow} & \downarrow \\ \mathcal{C}_0 & \xhookrightarrow[\text{id}]{} & \mathcal{C}_1 \end{array}$$

The fact that $\mathcal{C}_0 \xrightarrow{\text{id}} \mathcal{C}_1$ will be important when proving that $\text{Id}(x, y)$ always embeds into $\text{Iso}(x, y)$. For now, we will only use the fact that $\mathcal{C}_{\text{iso}}$ embeds into $\mathcal{C}_1$. Using this fact, we are back to define $\text{Iso}(x, y)$ as follows.

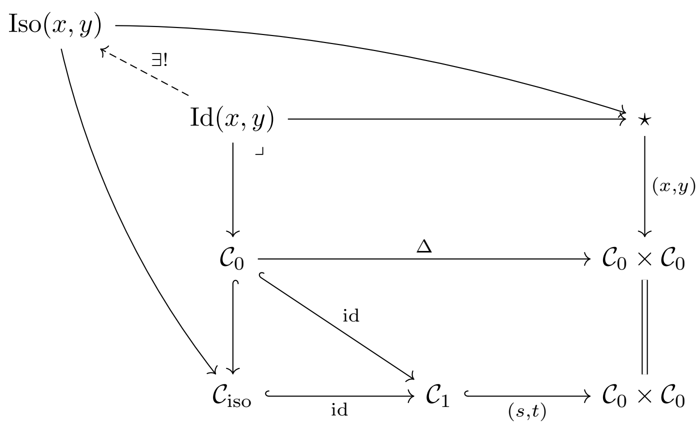
Definition 2.8. Let $\mathcal{C}$ be a category. The type $\text{Iso}(x, y)$ is defined by the following homotopy pullback square.

$$\begin{array}{ccccc} \text{Iso}(x,y) & & \dashrightarrow & & \star \\ \downarrow & \lrcorner\ \text{Ho} & & & \downarrow{\scriptstyle (x,y)} \\ \mathcal{C}_{\text{iso}} & \hookrightarrow & \mathcal{C}_1 & \xrightarrow[(s,t)]{} & \mathcal{C}_0 \times \mathcal{C}_0 \end{array}$$

Remark 2.9. *I could not find a source for this definition. Presumably, it is somewhere in [10], though this would probably require reading most of the book to make sense of their definition. The concept is mentioned on nlab, but they never give a definition nor a construction. I came to this working definition, based on discussions with Daniel Teixeira and Theo Johnson-Freyd.*

Intuitively, this pullback picks out isomorphisms $f : a \to b$ in $\mathcal{C}_{\text{iso}}$ such that $a \approx x$ and $b \approx y$. In general, not every equality is an isomorphism, so we would not expect to have an embedding of $\text{Iso}(x, y)$ into $\text{Id}(x, y)$ in general. However, the existence of identity isomorphisms should give rise to an embedding of $\text{Id}(x, y)$ into $\text{Iso}(x, y)$. In particular, the factorization of $\mathcal{C}_0 \xrightarrow{\text{id}} \mathcal{C}_1$ through $\mathcal{C}_{\text{iso}}$

can be used to fit $\mathrm{Id}(x, y)$ into the homotopy pullback square for $\mathrm{Iso}(x, y)$.



Then the univalence axiom states that for each pair of objects x and y, the inclusion of the identity type $\mathrm{Id}(x, y)$ into the isomorphism type $\mathrm{Iso}(x, y)$ is in fact an isomorphism.

TODO: The construction of $\mathrm{Id}(x, y)$, $\mathrm{Iso}(x, y)$, and $\mathrm{Id}(x, y) \to \mathrm{Iso}(x, y)$ are done somewhat informally. It would be nice to expand these details a bit more. This would require working out the construction of $\mathcal{C}_{\text{iso}}$.

TODO: It would be nice to find a tractable example of a univalent category that is not strict, or a category that is neither strict nor univalent.

TODO: It would be nice to look at the univalent completion of a category. This seems to require unpacking more details, such as the construction of $\mathcal{C}_{\text{iso}}$.

TODO: The HoTT book claims that their construction of **Set** is inherently univalent. They seem to define **Set** to be any category which satisfies Lawvere's list of axioms for the category of **Set** [6]. I guess these axioms admit both valent and univalent models, whereas the traditional axiomization seem to preclude univalence? This example might be illuminating to work through, though it sounds like this could be a bit of a rabbit hole.

TODO: I didn't have a chance to work out how this relates to the work being carried out in Oriented Category Theory. It would be nice to go back to that paper, review the section on univalence, and then work out why they care. The topics I covered in these notes were recommended by Theo Johnson-Freyd, based on the contents of the paper, but I never had a chance to revisit the paper after working out these details.

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