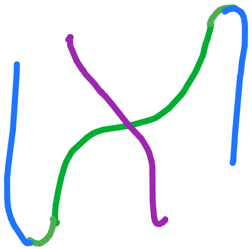

∞-categories w/ adjoints

are a bit like spaces

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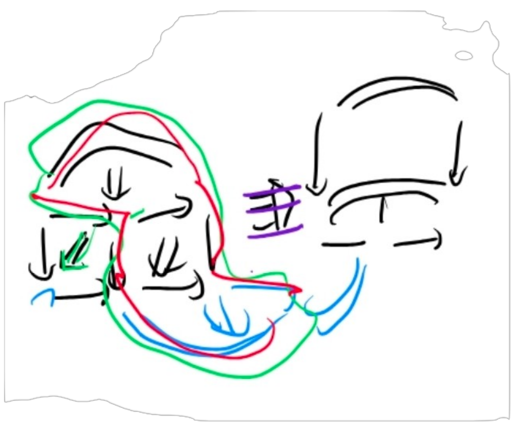
CMS 2026

$\cdots F^{LL} \dashv F^{L} \dashv F \dashv F^{R} \dashv F^{RR} \dashv \cdots$



Program: directed homotopy theory ~ (∞,∞)-categories

spaces ⟷ categories

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \mathbb{D}^1 \arrow[r] \arrow[ur, phantom, "\lrcorner"] & S^1 \end{tikzcd}$$


$$\begin{tikzcd} \partial\vec{\mathbb{D}}^1 \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \mathbb{D}^1 \arrow[r] \arrow[ur, phantom, "\lrcorner"] & \vec{S}^1 \end{tikzcd}$$


$\pi_1(X) = \mathrm{Map}_{S_{*}}(S^1,(X,x))/\sim$

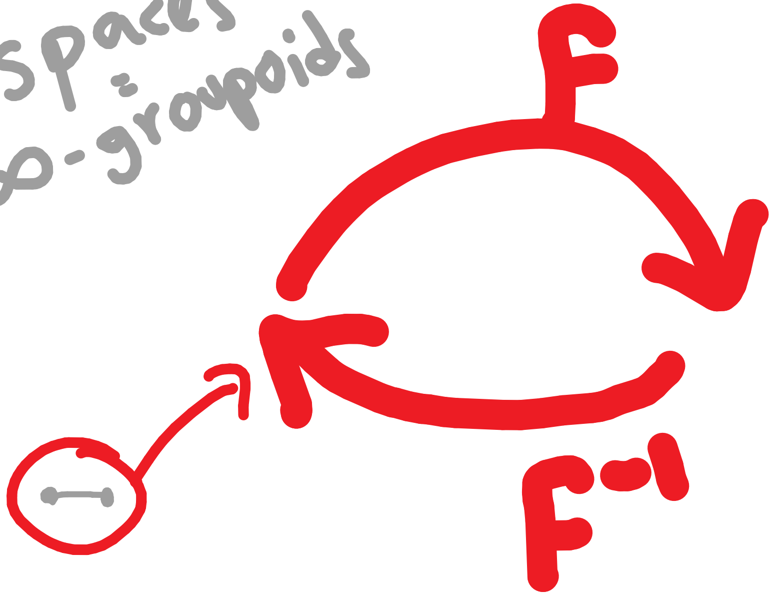
↳ loops

"π_1"$(\vec{S}^1,(C,c)) = \mathrm{Map}_{\mathrm{Cat}_*}(\vec{S}^1,(C,c))$

↳ directed loops

Program: directed homotopy theory ~ (∞,∞)-categories

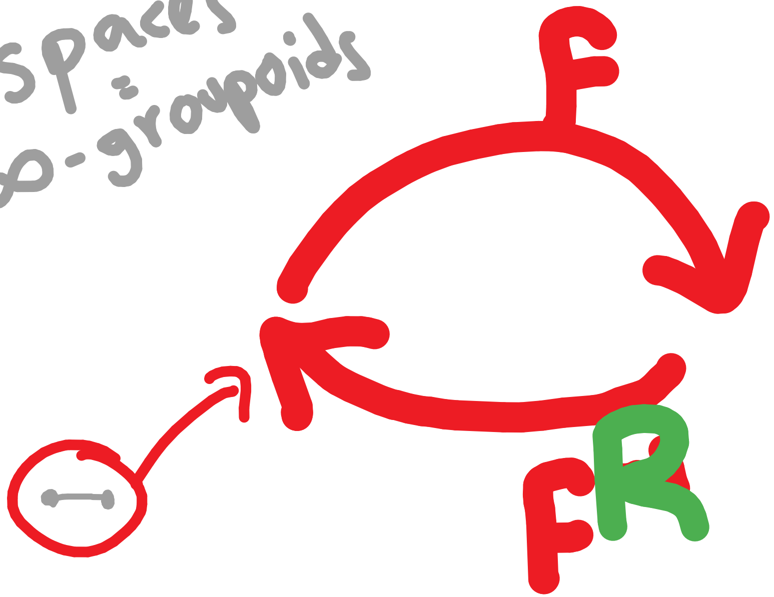
spaces = ∞-groupoids



$$ff^{-1} = id, \quad id = f^{-1}f$$

Program: directed homotopy theory $\sim (\infty,\infty)$-categories

spaces = ∞-groupoids



$FF^R \overset{\varepsilon}{\Rightarrow} \mathrm{id}, \quad \mathrm{id} \overset{\eta}{\Rightarrow} F^R F$

maybe?

- connections with dualizability
- TQFTs
- weak n-categories

Adjunctions in 2-categories

Def an adjunction in $\mathcal{C} \in 2\mathrm{Cat}$ =

$$
\left\{
\begin{tikzcd}
 & Y \arrow[rr, equal] \arrow[dr, "r"] & & Y \\
X \arrow[ur, "l"] \arrow[rr, equal] \arrow[r, phantom, shift left=3, "\Uparrow\eta"] & & X \arrow[ur, "l"'] \arrow[u, phantom, "\Uparrow\varepsilon"] &
\end{tikzcd}
=
\begin{tikzcd}
 & Y \arrow[rr, equal] & & Y \\
X \arrow[ur, "l"] \arrow[rr, equal] & & X \arrow[ur, "l"'] \arrow[ul, phantom, "\mathrm{id}_l"] &
\end{tikzcd}
\right.
$$

$$
\left.
\begin{tikzcd}
Y \arrow[rr, equal] \arrow[dr, "r"'] & & Y \arrow[dr, "r"] & \\
 & X \arrow[rr, equal] \arrow[ur, "l"'] \arrow[u, phantom, "\Uparrow\varepsilon"] & & X \arrow[ul, phantom, "\Uparrow\eta"]
\end{tikzcd}
=
\begin{tikzcd}
Y \arrow[rr, equal] \arrow[dr, "r"'] & & Y \arrow[dr, "r"] & \\
 & X \arrow[rr, equal] \arrow[ur, phantom, "\mathrm{id}_r"] & & X
\end{tikzcd}
\right\}
$$

Def adj = walking adjunction =

Rmk $\nearrow \xrightarrow{l \text{ or } r} \mathrm{adj}$ is epi...

rmk: I never use the word "pseudo"

Adjunctions in 2-categories

Def an adjunction in $\mathcal{C} \in 2\mathrm{Cat}$ =

$$\left\{\begin{tikzcd}
& Y \arrow[r, equal] \arrow[dr, "r"] \arrow[d, phantom, "\Uparrow\eta"] & Y \arrow[d, phantom, "\Uparrow\varepsilon"'] \\
X \arrow[ur, "l"] \arrow[r, equal] & X \arrow[ur, "l"'] &
\end{tikzcd}
=
\begin{tikzcd}
& Y \arrow[r, equal] & Y \\
X \arrow[ur, "l"] \arrow[r, equal] \arrow[urr, phantom, "\mathrm{id}_l"] & X \arrow[ur, "l"'] &
\end{tikzcd}
\quad
\begin{tikzcd}
Y \arrow[r, equal] \arrow[d, "r"'] \arrow[dr, phantom, "\Uparrow\varepsilon"] & Y \arrow[dr, "r"] \arrow[d, phantom, "\Uparrow\eta"] & \\
X & X \arrow[ul, "l"] \arrow[r, equal] & X
\end{tikzcd}
=
\begin{tikzcd}
Y \arrow[r, equal] \arrow[d, "r"'] \arrow[dr, phantom, "\mathrm{id}_r"] & Y \arrow[dr, "r"] & \\
X & X \arrow[r, equal] & X
\end{tikzcd}\right\}$$

Def adj = walking adjunction =

Rmk $\nearrow \xrightarrow{l \text{ or } r} \mathrm{adj}$ is epi

rmk: I never use the word "pseudo"

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Adjunctions in $(\infty,2)$-categories

Def? an adjunction in $\mathcal{C} \in (\infty,2)\mathrm{Cat}$ $=$

$$\left\{ \begin{array}{c}
\begin{tikzcd}
 & Y \arrow[rr, equal] \arrow[dr, "r"] & \Uparrow\varepsilon & Y \\
X \arrow[ur, "l"] \arrow[rr, equal] & \Uparrow\eta & X \arrow[ur, "l"'] &
\end{tikzcd}
\simeq
\begin{tikzcd}
 & Y \arrow[rr, equal] & \mathrm{id}_l & Y \\
X \arrow[ur, "l"] \arrow[rr, equal] & & X \arrow[ur, "l"'] &
\end{tikzcd}
\\
\begin{tikzcd}
Y \arrow[rr, equal] \arrow[dr, "r"'] & \Uparrow\varepsilon & Y \arrow[dr, "r"] & \\
 & X \arrow[ur, "l"] \arrow[rr, equal] & \Uparrow\eta & X
\end{tikzcd}
\simeq
\begin{tikzcd}
Y \arrow[rr, equal] \arrow[dr, "r"'] & \mathrm{id}_r & Y \arrow[dr, "r"] & \\
 & X \arrow[rr, equal] & & X
\end{tikzcd}
\end{array} \right\}$$

There is such a thing. But

Thm (Riehl-Verity, '16) $\mathrm{adj} \xrightarrow{\sim} \mathrm{hadj}$,

so we just use $\mathrm{adj} \in 2\mathrm{Cat} \hookrightarrow (\infty,2)\mathrm{Cat}$

Adjunctions in (∞,∞)-categories

1-morphisms

$$\begin{tikzcd} \to \arrow[r] \arrow[d] & \mathcal{C} \\ \mathrm{adj} \arrow[ur, dotted] & \end{tikzcd}$$

suspense

$$\infty\mathrm{Cat} \xrightarrow{\sigma} \infty\mathrm{Cat}$$
$$\mathcal{C} \longmapsto 0 \xrightarrow{\mathcal{C}} 1$$

ex $\mathcal{C} \in n\mathrm{Cat} \to \sigma\mathcal{C} \in (n+1)\mathrm{Cat}$

ex $\vec{\mathbb{D}}^{n} \rightsquigarrow \vec{\mathbb{D}}^{n+1} := \sigma\mathbb{D}^{n+1}$

$$\begin{tikzcd} \bullet \arrow[d] \\ \bullet \end{tikzcd} \rightsquigarrow \begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 1 \end{tikzcd}$$

n-morphisms

$$\begin{tikzcd} \sigma^{n-1}(\to) \arrow[r] \arrow[d] & \mathcal{C} \\ \sigma^{n-1}(\mathrm{adj}) \arrow[ur, dotted] & \end{tikzcd}$$

Adjunctions in (∞,∞)-categories

1-morphisms

$$\begin{tikzcd} \nearrow \arrow[r] \arrow[d] & \mathcal{C} \\ \mathrm{adj} \arrow[ur, dotted] & \end{tikzcd}$$

suspense

$$\infty\mathrm{Cat} \xrightarrow{\sigma} \infty\mathrm{Cat}$$
$$\mathcal{C} \longmapsto \bullet \xrightarrow{\mathcal{C}} 1$$

ex $\mathcal{C} \in n\mathrm{Cat} \to \sigma\mathcal{C} \in (n+1)\mathrm{Cat}$

ex $\vec{\Delta}^n \rightsquigarrow \vec{\Delta}^{n+1} := \sigma\vec{\Delta}^{n+1}$

$$\downarrow \;\rightsquigarrow\; \begin{tikzcd} 0 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 1 \end{tikzcd}$$

n-morphisms

$$\begin{tikzcd} \sigma^{n-1}(\nearrow) \arrow[r] \arrow[d] & \mathcal{C} \\ \sigma^{n-1}(\mathrm{adj}) \arrow[ur, dotted] & \end{tikzcd}$$

Adjunctions in (∞,∞)-categories

1-morphisms

$$\begin{tikzcd} \nearrow \arrow[r] \arrow[d] & \mathcal{C} \\ \mathrm{adj} \arrow[ur, dotted] & \end{tikzcd}$$

suspense

$$\begin{aligned} \infty\mathrm{Cat} &\xrightarrow{\sigma} \infty\mathrm{Cat} \\ \mathcal{C} &\longmapsto \bullet \xrightarrow{\mathcal{C}} 1 \end{aligned}$$

ex $\mathcal{C} \in n\mathrm{Cat} \to \sigma\mathcal{C} \in (n+1)\mathrm{Cat}$

ex $\vec{\mathbb{D}}^n \rightsquigarrow \vec{\mathbb{D}}^{n+1} := \sigma\mathbb{D}^{n+1}$

$$\downarrow \ \rightsquigarrow \ \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & 1 \end{tikzcd}$$

n-morphisms

$$\begin{tikzcd} \sigma^{n-1}(\nearrow) \arrow[r] \arrow[d] & \mathcal{C} \\ \sigma^{n-1}(\mathrm{adj}) \arrow[ur, dotted] & \end{tikzcd}$$

Yoga: reflective subcategories

Def $\mathcal{C}$ CAT, $S \subseteq \mathrm{Mor}\,\mathcal{C}$ $\to$ $X \in \mathcal{C}$ is $\underline{S\text{-local}}$ if

$$\begin{tikzcd} A \arrow[r] \arrow[d, "S \ni"'] & C \\ B \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

Prop $\mathcal{C}$ presentable + dom(S) compact

$$\Rightarrow \mathcal{C}_S \underset{L}{\overset{\perp}{\leftrightarrows}} \mathcal{C} \text{ is a reflective subcategory}$$

Sketch: $\hat{S}$ = codiag. compl. (S) $\xrightarrow{\text{small obj arg}}$ $(\mathrm{cell}(\hat{S}), \mathrm{urlp}(S)) \rightsquigarrow$ $X \xrightarrow{S\text{-loc}} \hat{X} \xrightarrow{\mathrm{urlp}(S)} \bullet$

Define $LX = \hat{X}$

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Yoga: reflective subcategories

$$\mathcal{G}pd \hookrightarrow \mathcal{C}at$$

$$\begin{tikzcd} \nearrow \arrow[r] \arrow[d] & \mathcal{G} \\ J \arrow[ur, dashed, "!"'] & \end{tikzcd}$$

$$\mathcal{G}aunt \hookrightarrow \mathcal{C}at$$

$$\begin{tikzcd} J \arrow[r] \arrow[d] & \mathcal{C} \\ \bullet \arrow[ur, dashed, "!"'] & \end{tikzcd}$$

$$(n,n)\mathcal{C}at \hookrightarrow n\mathcal{C}at$$

$$\begin{tikzcd} \sigma^{n+k}(S^1) \arrow[r] \arrow[d] & \mathcal{C} \\ \sigma^{n+k}(*) \arrow[ur, dashed, "!"'] & \end{tikzcd}$$

$$\mathcal{T}op_{\leq n} \hookrightarrow \mathcal{T}op$$

$$\begin{tikzcd} S^{\geq n} \arrow[r] \arrow[d, hook] & X \\ D^{\geq n+1} \arrow[ur, dashed, "!"'] & \end{tikzcd}$$

$$n\mathcal{C}at \hookrightarrow \infty\mathcal{C}at$$

$$\begin{tikzcd} \mathbb{D}^{n+k} \arrow[r] \arrow[d] & \mathcal{C} \\ \sigma^{n+k}(J) \arrow[ur, dashed, "!"'] & \end{tikzcd}$$

Yoga: reflective subcategories

Def $\infty\mathrm{CatAdj} \underset{\perp}{\overset{\hookrightarrow}{\leftarrow}} \infty\mathrm{Cat}$

$$\begin{tikzcd} \sigma^n(\nearrow) \arrow[r] \arrow[d] & \mathcal{C} \\ \sigma^n(\mathrm{adj}) \arrow[ur, dotted, "!"'] & \end{tikzcd}$$

automatic bc epi

$\{\vec{\mathbb{D}}^n\}$ is a conservative family

$\{L(\vec{\mathbb{D}}^n)$ is a conservative family$\}$

or: $\infty\mathrm{CatAdj} = \varprojlim\left(\cdots \to (n+1)\mathrm{CatAdj} \to (n\mathrm{CatAdj}) \to \cdots\right)$

or: $\infty\mathrm{CatAdj} = \varprojlim\left(\cdots \to \infty\mathrm{CatAdj}_n \to \infty\mathrm{CatAdj}_{n-1} \to \cdots\right)$

Yoga: reflective subcategories

"adjoining a lot of adjoints"

Def $\infty\mathrm{CatAdj} \underset{L}{\overset{}{\rightleftarrows}} \infty\mathrm{Cat}$, $L \dashv$

$$\begin{tikzcd} \sigma^n(\nearrow) \arrow[r] \arrow[d] & \mathcal{C} \\ \sigma^n(\mathrm{adj}) \arrow[ur, dotted, "!"'] & \end{tikzcd}$$

automatic bc epi

$\{\vec{\mathbb{D}}^n\}$ is a conservative family

$\{L(\vec{\mathbb{D}}^n)\}$ is a conservative family $\neq \sigma^n(\mathrm{adj})$

or: $\infty\mathrm{CatAdj} = \varprojlim(\cdots \to (n+1)\mathrm{CatAdj} \to (n\mathrm{CatAdj}) \to \cdots)$

or: $\infty\mathrm{CatAdj} = \varprojlim(\cdots \to \infty\mathrm{CatAdj}_n \to \infty\mathrm{CatAdj}_{n-1} \to \cdots)$

Some calculus, mate

Def

$$\alpha = \begin{tikzcd} \cdot \arrow[r, "l"] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, "l'"'] \arrow[ur, phantom, "\Nearrow"] & \cdot \end{tikzcd} \rightsquigarrow \begin{tikzcd} & \cdot \arrow[dl, "r"'] \arrow[d, equal] \\ \cdot \arrow[r] \arrow[d] \arrow[ur, phantom, "\Nearrow"] & \cdot \arrow[d] \\ \cdot \arrow[r] \arrow[ur, phantom, "\Nearrow"] & \cdot \arrow[dl, "r'"] \arrow[d, phantom] \\ \cdot \arrow[u, equal] \arrow[ur, phantom, "\Nearrow"] & \end{tikzcd} = \begin{tikzcd} \cdot \arrow[d] & \cdot \arrow[l] \arrow[d] \arrow[dl, phantom, "\Swarrow"] \\ \cdot & \cdot \arrow[l] \end{tikzcd} = \alpha^{\curvearrowright}$$

mate of α

via Gray products

Def α is <u>Beck-Chevalley</u> if
1. $\exists\, \alpha^{-1}$
2. $\exists\, (\alpha^{\curvearrowright})^{-1}$

this wk today ($B.C. = \vec{\mathbb{D}}' \otimes adj$)

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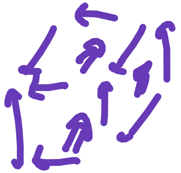
Some calculus, mate

$\mathbb{D}^0 = *$ $\quad$ $\mathbb{D}^1 = \longrightarrow$ $\quad$ $\mathbb{D}^1 \otimes \mathbb{D}^1 =$ $\begin{tikzcd} \bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$

$\downarrow \otimes \rightarrow$

$\mathbb{D}^{\otimes 3} =$  $\supseteq$ 

Thm (Campion) $\exists!$ closed monoidal structure extending this:

$- \otimes \mathcal{C} : \infty\mathrm{Cat} \rightleftarrows \infty\mathrm{Cat} : \mathrm{Fun}^{\mathrm{lax}}(\mathcal{C}, -)$ or $\mathcal{C} \otimes - \dashv \mathrm{Fun}^{\mathrm{oplax}}(\mathcal{C}, \mathcal{D})$

↳ really hard to control outside of certain polygraph inclusions

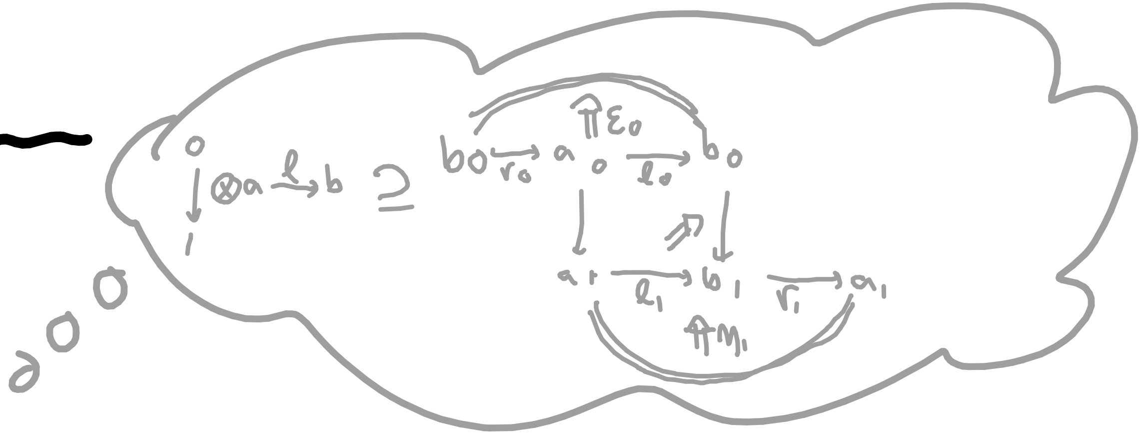
↳ just basic algebra here

e.g. $\mathbb{D}^2 \to \mathrm{Fun}^{\mathrm{lax}}(\vec{\mathbb{D}}^1, \mathcal{C}) \rightsquigarrow \mathbb{D}^2 \otimes \mathbb{D}^1 \to \mathcal{C} \rightsquigarrow$ 

modifications

Walking mate

$$\begin{tikzcd} \bullet \otimes \to \arrow[d, "\mathrm{id}\otimes \ell"'] \arrow[r, hook] & \downarrow\otimes\to \arrow[d, "\downarrow \ell"] \\ \bullet \otimes \mathrm{adj} \arrow[r] & \ulcorner \; \square \end{tikzcd} =: \mathrm{Mate}$$



Def $\downarrow\otimes\to \xrightarrow{\text{lax n.t.}} \infty\mathrm{Cat}$

$\downarrow\ell$

$\square \quad \text{lax}^{L}\text{ n.t.}$

Thm $\mathrm{LFun}^{\mathrm{lax}}(C,D) \simeq \mathrm{RFun}^{\mathrm{oplax}}(C,D)^{\mathrm{coop}}$

Rmk

$$\begin{tikzcd} \mathbb{D}' \arrow[r, "\simeq"] \arrow[d] & (\mathbb{D}')^{\mathrm{coop}} \arrow[d] \\ \mathrm{adj} \arrow[r, "\simeq"] & \mathrm{adj}^{\mathrm{coop}} \end{tikzcd} \Rightarrow \begin{tikzcd} \downarrow\otimes\to \arrow[r, "\simeq"] \arrow[d, "\ell"] & \downarrow\otimes\to^{\mathrm{coop}} \arrow[d, "r"] \\ \mathrm{Mate} \arrow[r] & \mathrm{Mate}^{\mathrm{coop}} \end{tikzcd}$$

Walking mate

Thm $\mathrm{LFun}^{\mathrm{lax}}(C,D) \simeq \mathrm{RFun}^{\mathrm{oplax}}(C,D)^{\mathrm{coop}}$

Conjecture (Masuda–Reutter) $\otimes|_{\infty\mathrm{CatAdj}}$ is symmetric

Not a proof but.

Prop. (T.) If $C, D \in \infty\mathrm{CatAdj}$, then $C \otimes D \simeq D \otimes C$.

Sketch: $\mathrm{LFun}^{\mathrm{lax}}(C,D) = \mathrm{Fun}^{\mathrm{lax}}(C,D) = \mathrm{Fun}^{\mathrm{oplax}}(C,D)^{1\mathrm{op},2\mathrm{op}}$

Fact: $\mathrm{Fun}^{\mathrm{lax}}(C,D) \in \mathrm{Adj}\infty\mathrm{Cat} \xleftrightarrow[\text{2-cells}]{\text{mate the}} \mathrm{Fun}^{\mathrm{oplax}}(C,D) = \mathrm{Fun}^{\mathrm{lax}}(C,D)^{2\mathrm{op},3\mathrm{op}}$

(no time srry

Walking mate

Thm: $LFun^{lax}(C,D) \simeq RFun^{oplax}(C,D)^{coop}$

Conjecture (Masuda-Reutter) $\otimes|_{\infty CatAdj}$ is symmetric

Not a proof but. $C \in \infty Cat$

Prop.(T.) If $D \in \infty CatAdj$, then $C \otimes D \simeq D \otimes C$.

Sketch: $LFun^{lax}(C,D) = Fun^{lax}(C,D) = Fun^{oplax}(C,D)^{1op,2op}$

Fact: $Fun^{lax}(C,D) \in Adj\infty Cat \xleftrightarrow[\text{2-cells}]{\text{mate the}} Fun^{oplax}(C,D) = Fun^{lax}(C,D)^{2op,3op}$

$-\otimes C \simeq C \otimes -$

(no time srry) $\rightsquigarrow Fun^{lax}(C,D) = Fun^{oplax}(C,D)^{n,n+1\,op}$ $\forall n$, take $n \to \infty \Rightarrow Fun^{lax}(C,D) = Fun^{oplax}(C,D)$

$\perp$ $\perp$

$\| / \|$

Walking mate

Thm $\mathrm{LFun}^{lax}(C,D) \simeq \mathrm{RFun}^{oplax}(C,D)^{coop}$

"Cor": B, C bicategories w/ adjoints $\longrightarrow B \otimes_{lax} C \simeq C \otimes_{lax} B$

↳ the technique works, but not the proof

Prop. (T.) If $C \in \infty Cat$, $D \in \infty CatAdj$, then $C \otimes D \simeq D \otimes C$.

Sketch: $\mathrm{LFun}^{lax}(C,D) = \mathrm{Fun}^{lax}(C,D) = \mathrm{Fun}^{oplax}(C,D)^{1op,2op}$

Fact: $\mathrm{Fun}^{lax}(C,D) \in \mathrm{Adj}\infty\mathrm{Cat} \underset{\text{remate the 1-cells}}{\overset{\text{mate the 2-cells}}{\rightleftarrows}} \mathrm{Fun}^{oplax}(C,D) = \mathrm{Fun}^{lax}(C,D)^{2op,3op}$

$-\otimes C \simeq C \otimes -$

no time srry

$\Rightarrow \mathrm{Fun}^{lax}(C,D) = \mathrm{Fun}^{oplax}(C,D)^{n,n+1\,op}$ $\forall n$, take $n \to \infty \Rightarrow \mathrm{Fun}^{lax}(C,D) = \mathrm{Fun}^{oplax}(C,D)$

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Walking mate

$\boxed{\text{Thm}}$ $\mathrm{LFun}^{lax}(C,D) \simeq \mathrm{RFun}^{oplax}(C,D)^{coop}$

"Cor": B, C bicategories w/ adjoints $\longrightarrow B \otimes_{lax} C \simeq C \otimes_{lax} B$

$\boxed{\text{Prop.(T.)}}$ If $C \in \infty\mathrm{CatAdj}$, $D \in \infty\mathrm{CatAdj}$, then $C \otimes D \simeq D \otimes C$.

TY.

Sketch: $\mathrm{LFun}^{lax}(C,D) = \mathrm{Fun}^{lax}(C,D) = \mathrm{Fun}^{oplax}(C,D)^{1op,2op}$

Fact: $\mathrm{Fun}^{lax}(C,D) \in \mathrm{Adj}\infty\mathrm{Cat} \xleftrightarrow[\text{remate the 1-cells}]{\text{mate the 2-cells}} \mathrm{Fun}^{oplax}(C,D) = \mathrm{Fun}^{lax}(C,D)^{2op,3op}$

(no time srry) $\Rightarrow \mathrm{Fun}^{lax}(C,D) = \mathrm{Fun}^{oplax}(C,D)^{n,n+1\,op}$ $\forall n$, take $n \to \infty \Rightarrow \mathrm{Fun}^{lax}(C,D) = \mathrm{Fun}^{oplax}(C,D)$

$-\otimes C \simeq C \otimes -$

||/||