(∞,∞)-categories w/ adjoints are a bit like spaces

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Simons Collaboration on Global Categorical Symmetries

∞-CATS W/ ADJOINTS

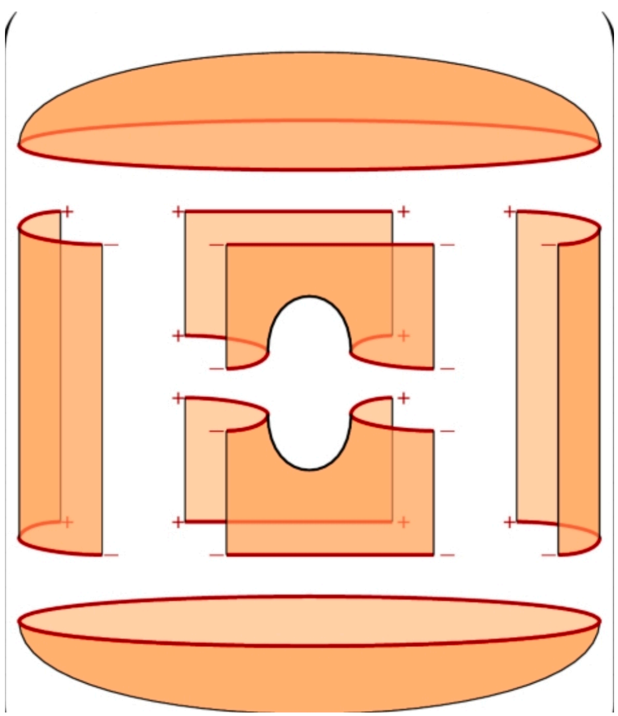
∞-CATS

SPACES

invertibility

TODAY'S GUIDING THREAD: the Cobordism Hypothesis
AKA
a universal property of bordism

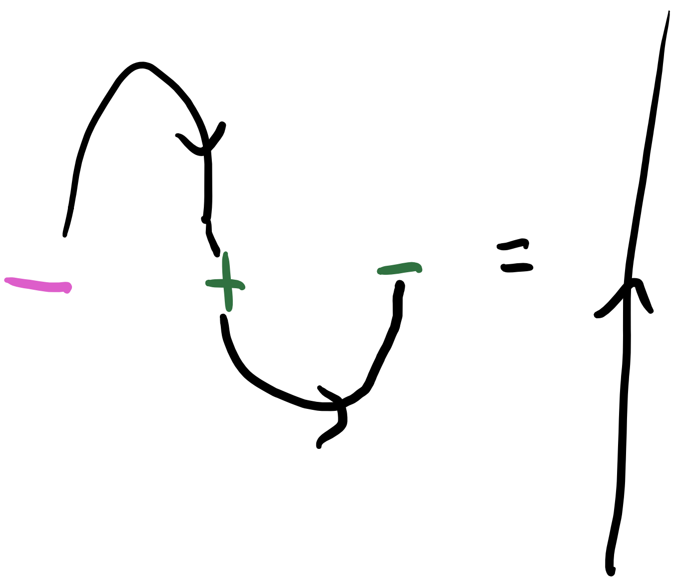
$\boxed{\text{Thm}}$ (C.H.) Bord_n^{fr} is the free symmetric (∞,n)-category w/ duals.

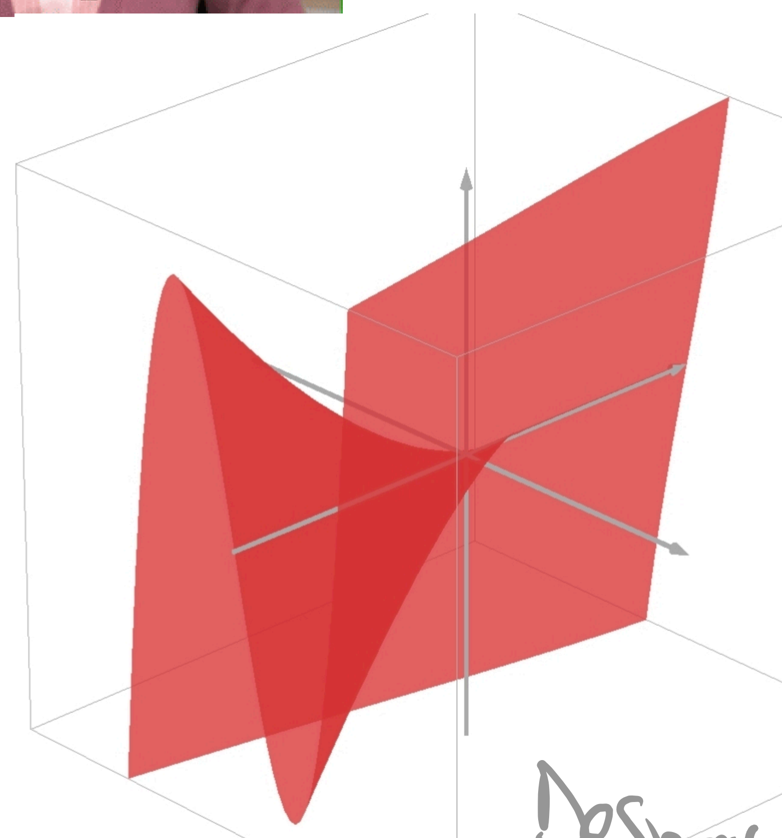


= a 2-morphism in Bord_2

Brunner-Carqueville-Rogenkamp, 23'

dim = 1 (Bord_1)





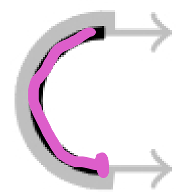
Desmos

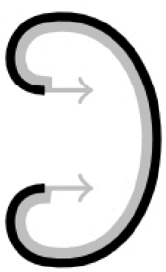
$\dim = 2$ $\to$ 2-dualizability

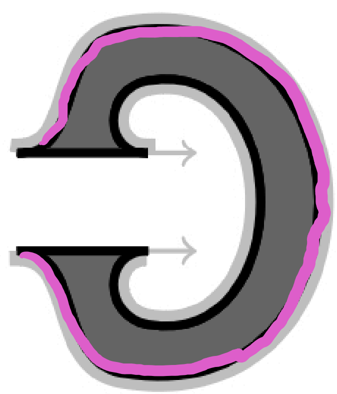
Bord_2

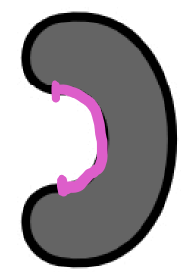
$\downarrow$

morphisms have adjoints

$\mathrm{ev}^L :=$ 

$\mathrm{ev}^R =$ 

$u_2 :=$ 

$v_2 :=$ 

Douglas–Schommer-Pries–Snyder

An n-category is roughly the following data:

+ composition

→ a set of objects

→ a set of morphisms btw each pair of objects

→ " 2-morphisms " of morphisms

⋮

→ a set of n-morphisms btw each pair of (n-1)-morphisms

An (∞, n)-category is ~roughly~ the following data:

+composition

→ a ~~set~~ space of objects

→ a ~~set~~ space of morphisms btw each pair of objects

→ " 2-morphisms " of morphisms

⋮

→ a ~~set~~ space of n-morphisms btw each pair of (n-1)-morphisms

① SPACES & ∞-CATEGORIES

Principle: our building blocks are homotopy types. $\rightsquigarrow$ = Kan complex

= ∞-groupoid

= anima.

"HOMOTOPY HYPOTHESIS"

Def: a category is:

- → a space of obj's
- → a space of morphisms btw each pair of obj's

+ compositions

TIMELINE:*

- 1900 — π_1
- H_n
- 1930 — π_n, H^n
- 1940
- 1945 — CATEGORIES
- 1950 — BG
- 1958 — SPECTRA
- 1960
- MODEL CATS
- E_n-OPERADS
- 1970
- 1973 — WEAK KAN COMPLEXES
- 1980

ALGEBRAIC TOPOLOGY

HOMOTOPY THEORY

CATEGORY THEORY

* not historically accurate

- 1985 — BRAIDED CATS
- 1990
- 1995 — COBORDISM HYPOTHESIS
- 2000
- ∞-CATEGORIES
- 2006 — HTT
- PROOF OF C.H.
- 2009 — HA
- 2010
- MODELS OF n-CATS
- 2020 ...

HOMOTOP. CAT. THRY

$$n\text{Cat} = \begin{cases} \text{a space of obj's} \to \text{Ob} \\ \text{an } (n-1)\text{-cat btw each } x, y \in \text{Ob} \end{cases}$$

$\boxed{\text{Def}}$

$$\infty\text{Cat} = \lim_{\leftarrow} \left(\cdots \to 2\text{Cat} \to 1\text{Cat} \to 0\text{Cat} \right)$$

$$0\text{Cat} = S$$

DOING HOMOTOPY(*) W/ ∞-CATEGORIES

→ products

→ disks

→ spheres

→ wedge

→ loops

→ suspension

(*) see recent work of Gepner, Hoine & Masuda

② SPECTRA & MONOIDAL CATEGORIES

$$\begin{array}{ccccc} A & = & \Omega(BA) & = & \Omega^2(B^2A) \\ \cap & & \cap & & \cap \\ \mathrm{CMon} & & \mathrm{Cat}_* & & 2\mathrm{Cat}_* \end{array}$$

② SPECTRA & MONOIDAL CATEGORIES

$\in$ E_2-spaces (e.g. $\bigsqcup B\,\mathrm{Braid}_n$)

$$A = \Omega(BA) = \Omega^2(B^2A)$$

$$\cancel{\cap} \qquad\qquad \cap \qquad\qquad \cap$$

$$\mathrm{CMon}(S) \qquad \mathrm{Cat}_* \qquad 2\mathrm{Cat}_*$$

② SPECTRA & MONOIDAL CATEGORIES

$$A = \Omega(BA) = \Omega^2(B^2A) = \Omega^3(B^3A) = \dots$$

$$\underset{\mathrm{CMon}(S)}{\cap} \qquad \underset{\mathrm{Cat}_\otimes}{\cap} \qquad \underset{2\mathrm{Cat}_\otimes}{\cap}$$

$\downarrow$

<u>DEF</u> (Lima, 1958): Ω-spectrum

SYMMETRIC
MONOIDAL
N-CATEGORIES
ARE CATEGORICAL
SPECTRA!

DOING STABLE HOMOTOPY W/ CATEGORICAL SPECTRA

→ SMASH

→ DOLD-THOM

→ DOLD-KAN?

→ SSEQS?

→ BROWN REPRESENTABILITY

→ CHAIN HOMOLOGY?

→ WHITE~~HEAD~~

→ CELLULAR HOMOLOGY?

*see recent work of Gepner, Haine & Masuda

Proof of the CH (Lurie)

induction: $\begin{cases} \text{base} = \mathrm{Bord}_1 \\ \text{inductive step:} \end{cases}$

$$\begin{tikzcd} \mathrm{Bord}_{n-1} \arrow[r, hook] \arrow[dr] & \mathrm{Bord}_n \arrow[d, dashed] \\ & \mathcal{C} \end{tikzcd}$$

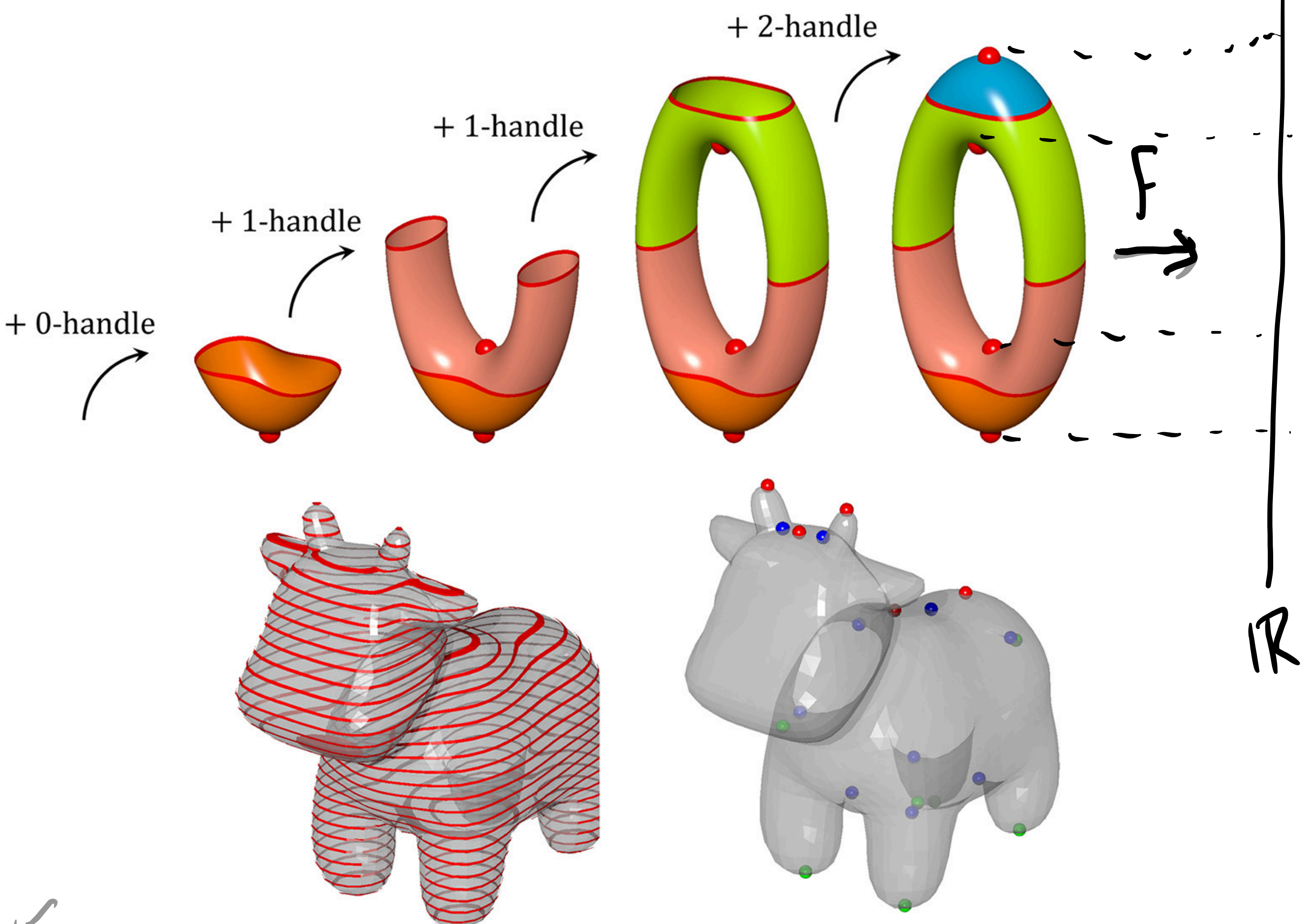
(i) $\mathrm{Bord}_n = \mathrm{Bord}_n^{+\text{framed function}}$

filter by Morse index

(ii) $\mathrm{Bord}_{n-1} \hookrightarrow F_1 \hookrightarrow F_2 \hookrightarrow \cdots \hookrightarrow F_n = \mathrm{Bord}_n^{+\text{framed function}}$

Why $\mathrm{Bord}_n = \mathrm{Bord}_n$

+Framed function

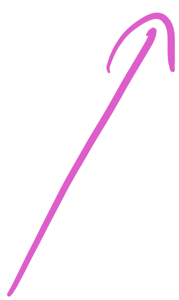


Pareja-Corcho et al, 2023

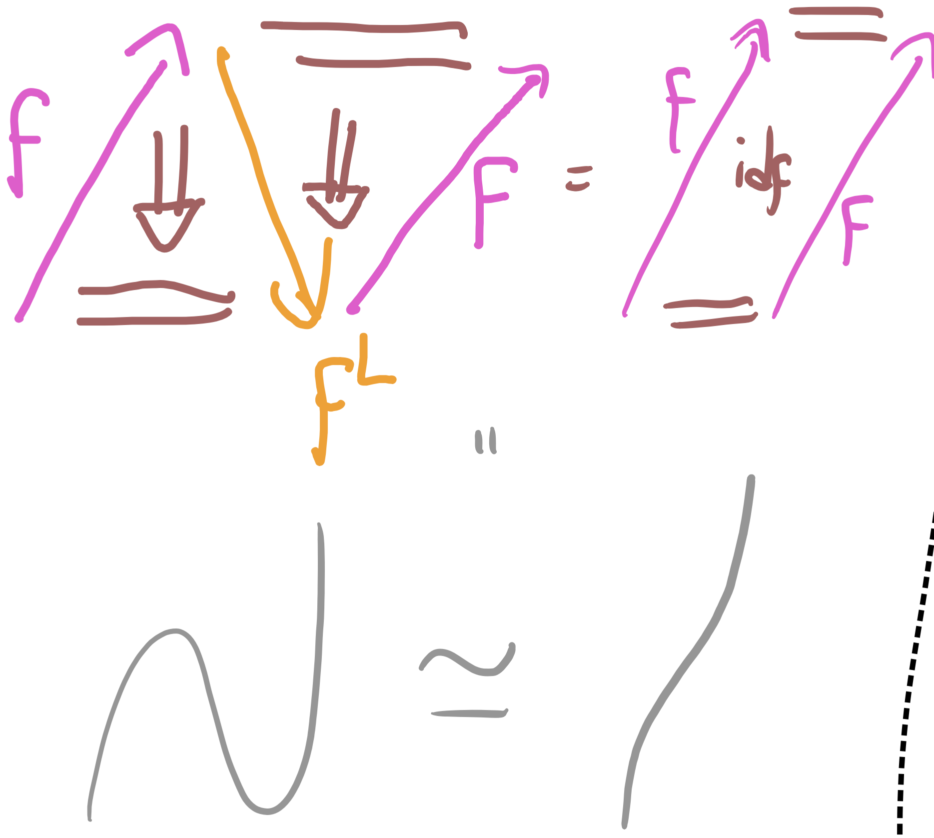
③ CATEGORIES W/ DUALS
& WHAT I DO W/ MY TIME

→ CATS

arrows

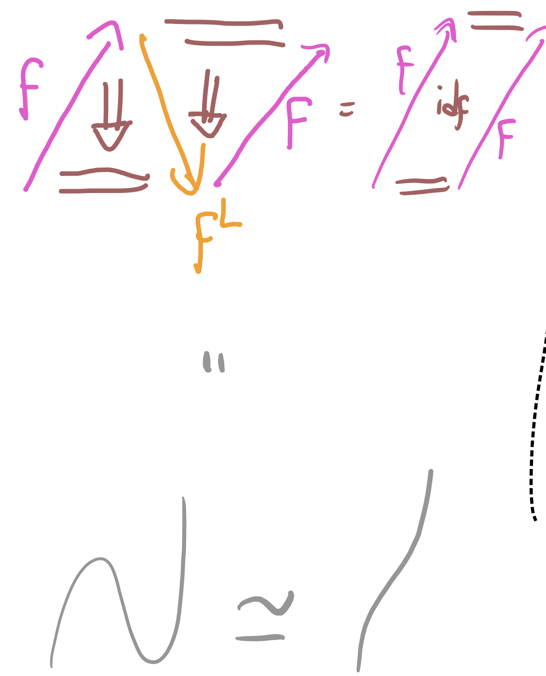


adjoints



→ SPACES

isos

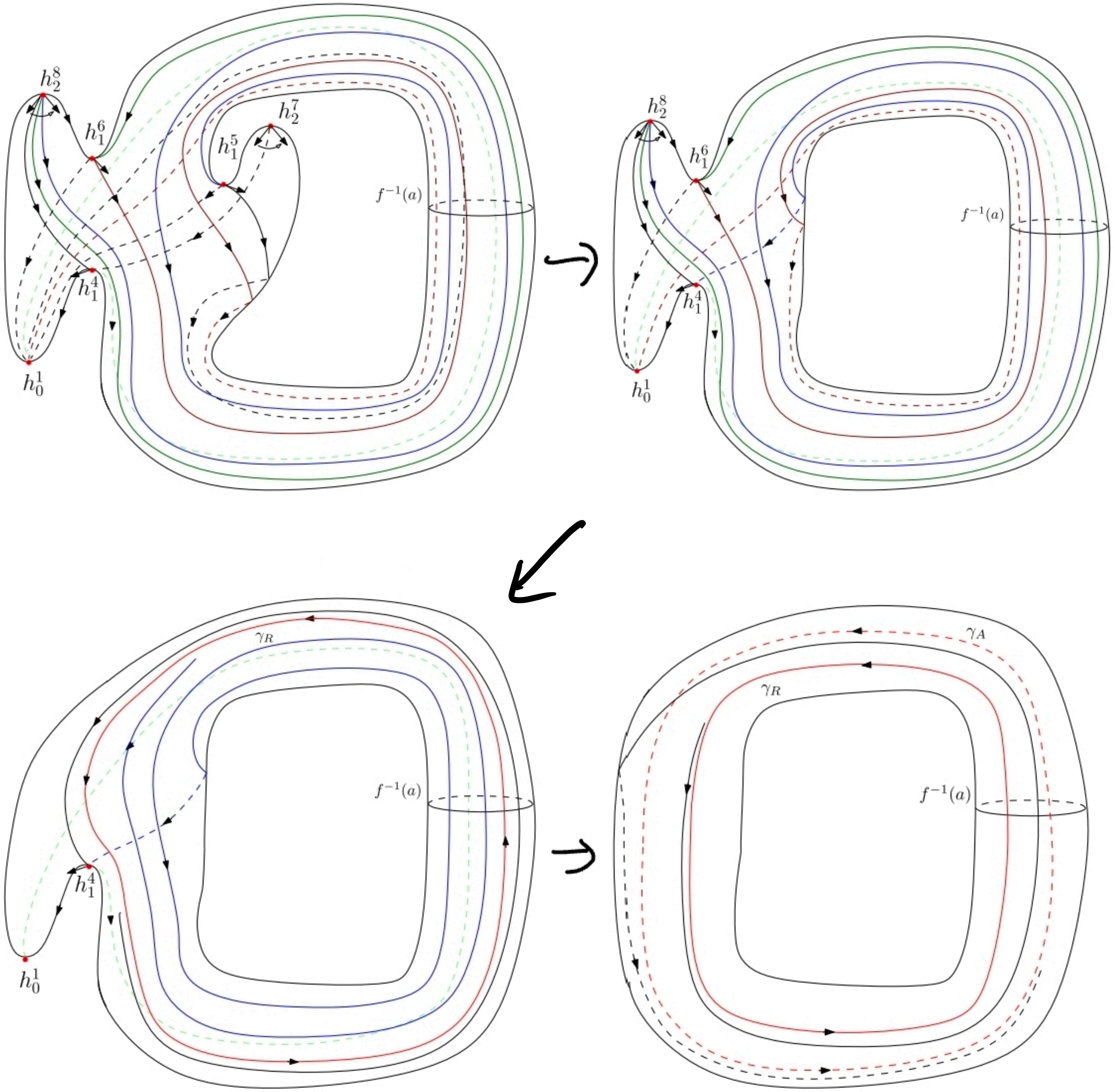


Def An n-category **has adjoints** if every $(<n)$-morphism has both adjoints.

SPACES	∞-CATS	n-CATS W/ADJ
$S^1 = B\mathbb{Z}$	$\vec{S}^1 = B\mathbb{N}$	Bord_1
$\mathbb{S}$	$\mathrm{FinSet}_{\sqcup}$	Bord_n^{F}
$\lvert \mathrm{Conf}_{\mathbb{R}^k}(\infty) \rvert = \Omega^k S^k$	$\mathrm{Conf}_{\mathbb{R}^k}(\bullet) = \Omega^k \vec{S}^k$	$\mathrm{Tang}_{n,k}$
∞-loop spaces	SymMonCat	SymMonCat w/duals

↳ SymMonCat w/inverses

Why $\mathrm{Bord}_n = \mathrm{Bord}_n^{\text{frame function}}$ → Cerf + some data



Lima-Neto-Rezende-Silveira (2016)

HYPOTHESIS: the "homotopy theory" of ∞-cats w/adjoints is not that bad

SPACES	∞-CATS	n-CATS W/ADJ
$S^1 = B\mathbb{Z}$	$\vec{S}^1 = B\mathbb{N}$	Bord_1
$\mathbb{S}$	FinSet_*	Bord_n^F
$\lvert\mathrm{Conf}_{\mathbb{R}^k}(\bullet)\rvert = \Omega^k S^k$	$\mathrm{Conf}_{\mathbb{R}^k}(\bullet) = \Omega^k \vec{S}^k$	$\mathrm{Tang}_{n,k}$
∞-loop spaces	SymMonCat	SymMonCat w/duals

$\hookrightarrow$ SymMonCat w/inverses

TO-DOS

1. Translate the proof of the C.H. to the language of cat. spectra w/adjoints. What do you learn?
2. $\exists$ Postnikov theory for n-cats w/adjoints?
   ↳ my (in progress) thesis
3. Can we deepen the relation to singularity theory?

BONUS: degenerate singularities & coherence in cats

Lurie CH proof, section 3.4:

the generic behavior of several parameter families of functions on B. This is feasible for small numbers of parameters (and in fact this is sufficient for our purposes: we can use the methods of §3.5 to reduce to the problem of understanding the $(2,1)$-categories $\tau_{\leq 2}\,\mathcal{B}(M)$) using Thom's theory of *catastrophes*. However, it will be more convenient to address the issue in a different way, using Igusa's theory of *framed functions*.

The first seven elementary catastrophes

Name	Symbol	Crank	Codimension	Germ	Unfolding terms
Fold	A_2	1	1	$x^3 + y^2$	x
Cusp	$A_3^{\pm}$	1	2	$\pm x^4 + y^2$	x^2, y
Swallowtail	A_4	1	3	$x^5 + y^2$	x^3, x^2, x
Butterfly	$A_5^{\pm}$	1	4	$\pm x^6 + y^2$	x^4, x^3, x^2, x
Hyperbolic umbilic	D_4^{+}	2	3	$x^3 + xy^2$	$x^2 + y^2$, x, y
Elliptic umbilic	D_4^{-}	2	3	$x^3 - xy^2$	$x^2 + y^2$, x, y
Parabolic umbilic	D_5	2	4	$x^4 - xy^2$	x^2, y^2, x, y

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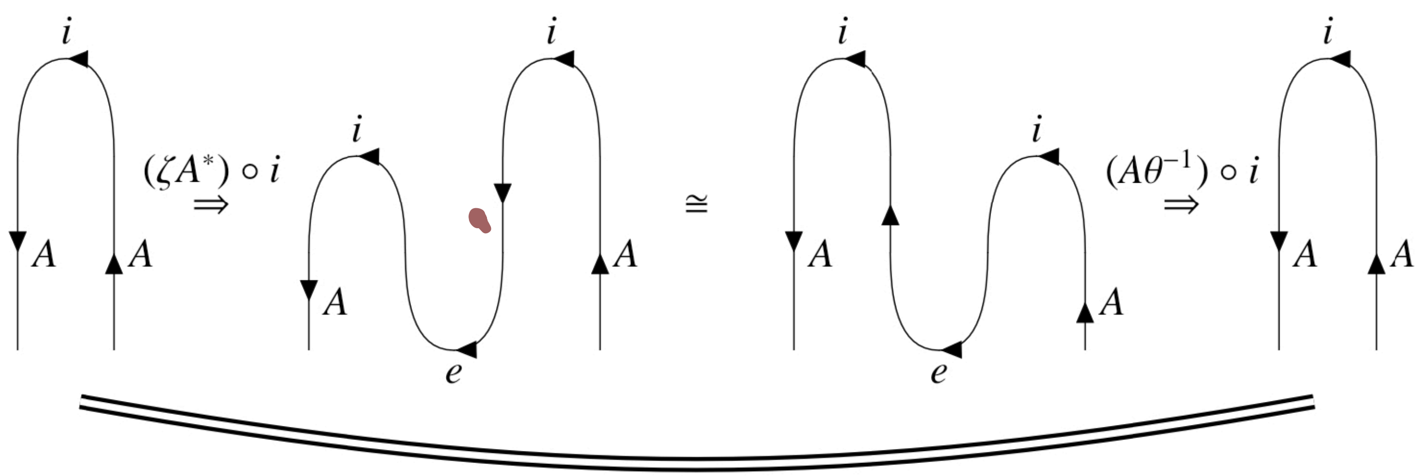
Butterfly	$A_5^{\pm}$	1			
Hyperbolic umbilic	D_4^{+}	2			
Elliptic umbilic	D_4^{-}	2			
Parabolic umbilic	D_5	2			

swallowtail equation



VALEU MUITO!