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Abstract

Concepts in spaces (=∞-groupoids) categorify verbatim to ω-categories once you have the correct definitions. We focus on defining n-truncated functors and showing that 1. for groupoids this is the same old topological notion, 2. the n-truncated categories are the n-categories.

Contents

We take the ∞-category of spaces $\mathcal{S}$ as a primitive notion.

1 Globes and spheres

The **n-dimensional globe** is the strict n-category Θ_n with two objects, two parallel k-morphisms for $0 < k < n$, and a single n-morphism. Write I for the $1d$ disk.

$$\Theta_0 = \ast\ , \qquad \Theta_1 = \ast \longrightarrow \ast \qquad \Theta_2 = \begin{tikzcd} \ast \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \ast \end{tikzcd} \qquad \dots$$

The **boundary of the n-globe** is given by removing the top dimensional morphism.

$$\partial\Theta_0 = \emptyset\ , \qquad \partial\Theta_1 = \ast \qquad \ast \qquad \partial\Theta_2 = \begin{tikzcd} \ast \arrow[r, bend left] \arrow[r, bend right] & \ast \end{tikzcd} \qquad \dots$$

The boundary of the next globe can be obtained from the previous ones by a pushout:

$$
\begin{tikzcd}
\partial\Theta^d \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \Theta^d \arrow[d] \\
\Theta^d \arrow[r] & \partial\Theta^{d+1}
\end{tikzcd}
$$

In contrast, the **directed n-sphere** is given by the following cofiber sequence:

$$
\begin{tikzcd}
\partial\Theta^d \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Theta^d \arrow[d] \\
* \arrow[r] & \vec{S}^n
\end{tikzcd}
$$

The reason why $\vec{S}^n \neq \partial\Theta^d$ is that $\Theta_n \neq *$, unlike for spaces / ∞-groupoids. In fact, the map $\Theta_n \to *$ is (≥ 0)-faithful, but (≤ 0)-full.

Likewise, $\vec{S}^n$ doesn't embed non-trivially in $\Theta^{d+1|}$: it does so in $\mathbb{J}^{d+1}$, the walking $(d+1)$-equivalence.

More elaborate constructions of homotopy theory categorify likewise, but in general there will be two options given by the choice of $\vec{S}^n$ vs $\partial\Theta_n$. For instance, loop spaces categorify to *loop categories*, i.e. $\Omega\mathcal{C} := \mathbf{Cat}_\omega(\vec{S}^1, \mathcal{C})$, and Ω-spectra to categorical spectra, i.e. a sequence $\mathfrak{C}^\bullet$ of ω-categories with $\mathfrak{C}^\bullet \cong \Omega\mathfrak{C}^{\bullet+1}$.

A more ellaborate categorification is that of suspension; one way to define it is as a left adjoint $\Sigma \dashv \Omega$; it has been shown by Masuda that $\Sigma\mathcal{C} \cong \vec{S}^1 \wedge \mathcal{C}$, where $\wedge$ is obtained from the *Gray* product (which categorifies Cartesian products).

A good reason to jump to $\mathbf{Cat}_\omega$ instead of containing yourself with n-categories is that these constructions are much cleaner as you don't have to keep track of several connectivity conditions. For instance we will push the following slogan throughout this note:

-n-truncated spaces are to spaces, as n-categories are to ω-categories-

In particular, $\Sigma \dashv \Omega$ respectively raise and decrease connectivity - in both senses! If you try to reason this with $\mathbf{Cat}_n$, you have to keep track of this raising and decreasing. So we work with $\mathbf{Cat}_\omega$.

2 From weak to orthogonal factorization systems

We denote a collection of morphisms in $\mathcal{C}$ by $S \subseteq \mathcal{C}$. This is not ambiguous as we don't have a concept of subcategory.

Definition 2.1. A **weak factorization system** on $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of collections of morphisms $\mathcal{L}, \mathcal{R} \subseteq \mathcal{C}$ such that

- every morphism $f \in \mathcal{C}$ factors as $f = r \circ l$, where $l \in \mathcal{L}$ and $r \in \mathcal{R}$;

- $\mathcal{R} = \mathrm{rlp}(\mathcal{L})$;
- $\mathcal{L} = \mathrm{llp}(\mathcal{R})$.

Locally presentable nonsense guarantees the existence of a weak factorization system $(\mathcal{L}, \mathrm{rlp}(S))$ for any $S \subseteq \mathcal{S}$. The following is the description of $\mathcal{L}$:

Definition 2.2. A morphism $A \to X$ is an **S-cell complex** if it is obtained by a transfinite composition of pushouts of pushouts with morphisms in S.

Denote the collection of S-cell complexes by $\mathrm{cell}(S)$.

Proposition 2.3. *The pair* $(\mathrm{cell}(S), \mathrm{rlp}(S))$ *is a weak factorization system.*

Proof. `sorry` □

The word "weak" contrasts with the following definitoin. There, the word "orthogonal" was historically added for clarity as w.f.s. gained importance due to model categories.

Definition 2.4. An **orthogonal factorization system** is a weak factorization whose factorizations are unique, or equivalently whose lifting properties are unique.

Our next goal is to upgrade $(\mathrm{cell}(S), \mathrm{rlp}(S))$ to an orthogonal factorization system whose right class is $\mathrm{orlp}(S)$.

Definition 2.5. The **codiagonal** of a morphism $f \in \mathcal{C}$ is the following morphism $\delta_f \in \mathcal{C}$:

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "f"'] & y \arrow[d] \arrow[ddr, bend left, equal] & \\
y \arrow[r] \arrow[drr, bend right, equal] & y \sqcup_x y \arrow[dr, dashed, "\delta_f"] & \\
& & b
\end{tikzcd}$$

A collection of morphisms $S \subseteq \mathcal{C}$ is **codiagonal complete** if S is closed under forming codiagonals.

Proposition 2.6. *A weak factorization system* $(\mathcal{L}, \mathcal{R})$ *is orthogonal iff* $\mathcal{L}$ *is codiagonal complete.*

Proof. `sorry` □

The *codiagonal completion* of a collection of morphisms $S \subseteq \mathcal{C}$ is the collection $\widehat{S} \subseteq \mathcal{C}$ obtained by adjoining codiagonals.

Corollary 2.7. *The pair* $(\mathrm{cell}(\widehat{S}), \mathrm{orlp}(S))$ *is an orthogonal factorization system for any* $S \subseteq \mathcal{C}$.

Proof. Combine Proposition 2.3 with Proposition 2.6. □

3 (n-connected, n-truncated) - spaces

Definition 3.1. A space $X \in \mathcal{S}_*$ is **n-connected** [resp. **n-truncated**] if $\pi_k(X) = 0$ for $k \leq n$ [resp. $k > n$].

Example 3.2.

- The n-sphere is $(n-1)$-connected.
- A simply connected space is 1-connected.
- Groupoids are 1-truncated.
- The Eilenberg-MacLane space $B^n G$ is $(n-1)$-connected and $(n-1)$-truncated.

$\triangle$

Let $(\partial D^{>n} \hookrightarrow D^{>n})$ [resp. $(\partial D^{\leq n} \hookrightarrow D^{\leq n})$] denote the set of boundary inclusions $\partial D^d \hookrightarrow D^d$ for $k > n$ [resp. $k \leq n$]. Then the following is trivial.

Proposition 3.3. *A space $X \in \mathcal{S}_*$ is n-connected [resp. n-truncated] iff $X \in \mathrm{orlp}(\partial D^{>n+1} \hookrightarrow D^{>n+1})$ [resp. $X \in \mathrm{orlp}(\partial D^{\leq n+1} \hookrightarrow D^{\leq n+1})$].*

Proof. The only reason why is this confusing is because the degrees, as for instance

$$\pi_{>n}(X) = 0 \iff \mathrm{orlp}(S^{>n} \hookrightarrow D^{>n+1}) \iff \mathrm{orlp}(\partial D^{>n+1} \hookrightarrow D^{>n+1}),$$

and similarly for n-truncated spaces. $\square$

This suggests the following definition:

Definition 3.4. A map $f : X \to Y$ in $\mathcal{S}_*$ is **n-connected** [resp. **n-truncated**] if $f \in \mathrm{orlp}(\partial D^{\leq n} \hookrightarrow D^{\leq n})$ [resp. $f \in \mathrm{orlp}(\partial D^{>n} \hookrightarrow D^{>n})$].

Remark 3.5. A *space* is n-connected if the terminal *map* is $(n+1)$-connected. 😕

By Corollary 2.7 there exists an orthogonal factorization system $(\mathcal{L}, \text{n-truncated})$, where $\mathcal{L} = \mathrm{cell}(\widehat{D}^{>n})$. The following proposition identifies $\mathcal{L}$ with the n-connected maps.

Proposition 3.6. *The pair $(n\text{-connected}, n\text{-truncated})$ is an orthogonal factorization system in $\mathcal{S}$.*

Proof. `sorry` $\square$

The following is the usual textbook definition of an n-connected morphism. identification of n-truncated maps with $\mathrm{cell}(\partial D^{>n+1} \hookrightarrow D^{>n+1})$ explains the following textbook definition.

Proposition 3.7. *A map is n-connected iff it induces an isomorphism on π_k in degree $k < n+1$ and an epimorphism in degree $n+1$.*

Proof. Lifting against $\partial D^d \hookrightarrow D^d$ for $d < n+1$ gives the isomorphisms in degree $< n+1$. Codiagonal completing gives no additional lifting conditions, except for that against $\partial D^{n+2} \twoheadrightarrow D^{n+1}$, which corresponds to an epimorphism in π_n. ☐

For n-truncated spaces there is no surprise:

Proposition 3.8. *A map is n-truncated iff it induces isomorphisms on π_k*

Also note the following:

Proposition 3.9. *A space X is n-connected iff X is an E^n-space.*

Proof. sorry ☐

Let $i : \mathcal{S}^{\leq n} \hookrightarrow \mathcal{S}$ denote the fullsubcategory on n-truncated spaces.

Proposition 3.10. *i is a reflective localization*

Proof. It is precisely the localization at $(\partial D^{>n} \hookrightarrow D^{>n})$.

sorry ☐

4 (n-connected, n-truncated) - categories

Let $\mathbf{Cat}_0 := \mathcal{S}$ and define $\mathbf{Cat}_{n+1} := \mathbf{Cat}_n\text{-}\mathbf{Cat}$. Some abstract nonsense guarantees that there is an inclusion functor $i : \mathbf{Cat}_n \to \mathbf{Cat}_{n+1}$. More nonsense guarantees that i is a (composite of) left adjoint(s); in the next section we will show that it is a reflective subcategory by finding an explicit localizing set. Define $\mathbf{Cat}_\omega$ as the direct limit of the inclusions in $\mathbf{Pr}^R$.

Recycling notation, let $\Theta^{>n}$ [resp. $\Theta^{\leq n}$] denote the sets of boundary inclusions of globes $\partial\Theta^k \hookrightarrow \Theta^k$ for $k > n$ [resp $k \leq n$]. We apply Corollary 2.7 to obtain a factorization system in $\mathbf{Cat}_\infty$.

Proposition 4.1. *The pair*

$$(\mathrm{cell}(\widehat{\partial\Theta^{>n} \hookrightarrow \Theta^{>n}}), \mathrm{orlp}(\partial\Theta^{>n} \hookrightarrow \Theta^{>n}))$$

is an orthogonal factorization system in $\mathbf{Cat}_\omega$.

We then mimick the definitions for spaces:

Definition 4.2. A functor $F : \mathcal{C} \to \mathcal{D}$ in $\mathbf{Cat}_\omega$ is **n-connected** [resp. **n-truncated**] if $f \in \mathrm{orlp}(\partial\Theta^{\leq n} \hookrightarrow \Theta^{\leq n})$ [resp. $f \in \mathrm{orlp}(\partial\Theta^{>n} \hookrightarrow \Theta^{>n})$].

Proposition 4.3. *A functor is n-truncated iff it is $(> n)$-fully faithful.*

Proof. Stare at the following diagram for long enough to conclude that the orlp against $\partial\Theta^d \hookrightarrow \Theta^d$ is d-fully-faithfulness:

$$\begin{tikzcd} \partial\Theta^d \arrow[r] \arrow[d] & \mathcal{C} \\ \Theta^d \arrow[ur, dashed, "\exists!"'] & \end{tikzcd} .$$

Taking the range $d > n$ yields the result. $\square$

Definition 4.4. A category $\mathcal{C} \in \mathbf{Cat}_\omega$ is **n-connected** [resp. **n-truncated**] if the terminal map is $(n+1)$-connected [resp. $(n+1)$-truncated].

Corollary 4.5. *A category is n-truncated iff it is an n-category.*

Proof. The terminal map is $(\geq n)$-fully faithful iff the space of $(\geq n)$-morphisms are trivial. $\square$

Also note the relation to monoidality:

Proposition 4.6. *A category $\mathcal{C}$ is n-connected iff it is a n-tuply monoidal (∞, ∞)-category.*

Proof. This is the delooping hypothesis. $\square$

Proposition 4.7. *A functor $F : \mathcal{C} \to \mathcal{D}$ is n-connected iff it is $(\leq n)$-faithful and $(< n)$-full.*

Proof. Lifting against $\partial\Theta^d \hookrightarrow \Theta^d$ for $d < n$ gives $(< n)$-fully-faithfulness. Codiagonal completing gives no additional lifting conditions, except for that against $\partial\Theta^{n+1} \twoheadrightarrow \Theta^n$, which corresponds to n-faithfulness. $\square$

Question 4.8. *The same abstract nonsense gives you a factorization system $(\mathcal{L}, n\text{-connected})$, where $\mathcal{L} = \mathrm{cell}(\Theta^{<n}$. Is there an alternative description of $\mathcal{L}$?*

The (n-connected, n-truncated)-factorization system passes to the reflective subcategories $\mathbf{Cat}_n \hookrightarrow \mathbf{Cat}_\omega$ via the following lemma.

Lemma 4.9. *Let $i : \mathcal{D} \hookrightarrow \mathcal{C} : L$ be a reflective subcategory and consider the wfs $(\mathrm{cell}(S), \mathrm{rlp}(S)))$ defined by $S \subseteq \mathcal{C}$. Then $(\mathrm{cell}(L(S)), \mathrm{rlp}(L(S)))$ is a wfs in $\mathcal{D}$.*

Proof. The operations defining $\mathrm{cell}(S)$ are all colimit operations, so L commutes with cell. A factorization system with $\mathrm{cell}(L(S))$ on the left exists by Proposition 2.3, and the right class must be $\mathrm{rlp}(L(S))$ since it is defined by the left calssclass. $\square$

Corollary 4.10. *In the situation of Lemma 4.9, if $(\mathrm{cell}(S), \mathrm{rlp}(S))$ is orthogonal then so is $(\mathrm{cell}(L(S)), \mathrm{rlp}(L(S)))$.*

Proof. This follows from Proposition 2.6 since codiagonals are colimit operations so L preserves them. $\square$

Let $L : \mathbf{Cat}_\omega \to \mathbf{Cat}_n$ be the left adjoint to inclusion. We will see at the end of this section that this is a reflective localization.

Corollary 4.11. *The pair*

$$(\mathrm{cell}(L(\widehat{\partial\Theta^{>n} \hookrightarrow \Theta^{>n}})), \mathrm{orlp}(L\partial\Theta^{>n} \hookrightarrow \Theta^{>n}))$$

is an orthogonal factorization system in $\mathbf{Cat}_n$.

The following 4 examples are obtained by applying **??** to **Cat**.

Example 4.12 (all,eso+ff)**.** The orlp against

$$\partial\Theta^0 \hookrightarrow \Theta^0 \quad \text{and} \quad \partial\Theta^1 \hookrightarrow \Theta^1$$

corresponds to surjectivity and fully-faithfulness, respectively. This already characterizes the right class. △

Example 4.13 (eso,ff)**.** The orlp against just $\partial\Theta^1 \hookrightarrow \Theta^1$ is fully-faithfulness; this already guarantees lifts against higher cells. There is a well-known orthogonal factorization system (eso, ff) in **Cat**, whence we conclude that eso $=$ cell$(\partial\Theta^1 \hookrightarrow \Theta^1)$. △

Example 4.14 (eso+f,f)**.** The globe $\partial\Theta_2 \hookrightarrow \Theta_2$ is localized to $\partial\Theta_2 \twoheadrightarrow \Theta_1$ in **Cat**. The orlp against this functor corresponds to faithfulness, and with the well-known orthogonal factorization system (eso+f, f) we conclude that (eso+f) $=$ cell$(\partial\Theta_2 \twoheadrightarrow \Theta_1)$. △

Example 4.15 (eso+ff,all)**.** The orlp against $L(\partial\Theta^d \hookrightarrow \Theta^d)$ doesn't do anything in **Cat** for $n > 3$. △

Example 4.16. **??** gives 5 interesting factorization systems in $\mathbf{Cat}_2$:

- (all,bieq)
- (eso,locally eso+locally ff)
- (eso+locally eso,ff)
- (eso+locally eso+f,f)
- (bieq,all)

△

Proposition 4.17. *The left adjoint $L : \mathbf{Cat}_\omega \to \mathbf{Cat}_n$ is precisely the localization of n-truncated morphisms.*

Proof. It is the localization at $(\partial\Theta^{>n} \hookrightarrow \Theta^{>n})$. `sorry` □

5 ((n-0.5)-connected,(n-0.5)-truncated) - categories

In the previous section we considered the orthogonal factorization systems generated by considering all boundary inclusions $\partial\Theta^d \hookrightarrow \Theta^{d+1}$. up to n, or greater than n. Orthogonality is obtained by codiagonal completion, which corresponded to adding the projection $\partial\Theta^{d+1} \twoheadrightarrow \Theta^d$. We could then consider the generating set which contains the latter but not the former (codiagonal completing it doesn't do anything as this is an epi).

Let $\Theta^{>n-0.5}$ denote $(\partial\Theta^{>n} \hookrightarrow \Theta^{>n})$ adjoined with the functor $\partial\Theta^n \twoheadrightarrow \Theta^n$, and define $\Theta^{\leq(n-0.5)}$ similarly.

Definition 5.1. A functor $F : \mathcal{C} \to \mathcal{D}$ in $\mathbf{Cat}_\omega$ is **(n-0.5)-connected** (resp. **(n-0.5)-truncated**) if $f \in \mathrm{orlp}(\Theta^{\leq (n-0.5)})$ (resp. $f \in \mathrm{orlp}(\Theta^{>(n-0.5)})$). A category $\mathcal{C} \in \mathbf{Cat}_\omega$ is **(n-0.5)-connected** (resp. **(n-0.5)-truncated**) if the terminal map is $(n-0.5)$-connected (resp. $(n-0.5)$-truncated).

Proposition 5.2. *A category $\mathcal{C} \in \mathbf{Cat}_\omega$ is (0.5)-connected iff it is a poset.*

Proof. We know by **??** $\square$

Proposition 5.3. *A functor is $(n-0.5)$-truncated iff it is $(\geq n)$-full and $(> n)$-faithful.*

Proof. `sorry` $\square$

Example 5.4. In **Cat** there exists an orthogonal factorization system whose right class is $\mathcal{R} = (eso + faithful)$. [1] A category is $\mathcal{R}$-local $\triangle$

6 walking n-isomorphisms

WARNING - THIS SECTION IS SKETCHY

Definition 6.1. The **walking n-isomorphism** is the n-category $\mathbb{I}^n$ obtained by inverting the top morphism in Θ^d. Also set $\mathbb{I}^0 := \Theta^0$.

Remark 6.2. Convince yourself that $\mathbb{I}^n \in \mathrm{cell}(\widehat{\Theta}^{\leq n})$.

Proposition 6.3. $\mathbb{I}^{n+1} \cong \Theta^n$

Proof. Contract the invertible arrow in $\mathbb{I}^{n+1}$. $\square$

We will apply this identification without mention. That means that we are really in a weak context for colimits.

Proposition 6.4. *Let $S \subseteq \mathbf{Cat}_1$ be comprised by $\partial\Theta^2 \hookrightarrow \Theta^1$ and $\partial\Theta^1 \twoheadrightarrow \Theta^0 \cong \mathbb{I}^1$. Then* $\mathrm{orlp}(S)$ *consists of the pseudomonic functors, i.e. faithful, and full on isomorphisms.*

Proof. Faithfulness correspond to lifts against the first functor, and unique lifts against the second mean that that every isomorphism in the image comes from a unique isomorphism in the domain. $\square$

Example 6.5. The terminal functor Θ_1 is in $\mathrm{orlp}(S)$. $\triangle$

Corollary 6.6. *There exists a factorization system in* **Cat** *whose right class are the pseudomonic functors.*

Question 6.7. *Is the right class the "pseudoepic" functors?*

RLP against $\Theta^d \hookrightarrow \mathbb{I}^n$ indicates that the n-morphisms are invertible. For n-categories,
RLP against $\mathbb{I}^n \to *$ indicates that there are no non-trivial n-isomorphisms.
RLP against $\vec{S}^n \hookrightarrow \mathbb{I}^n$ indicates that there are no non-trivial n-*auto*morphisms

[1] I don't know what is the left class.

Example 6.8. A category is orthogonal to $\Theta^1 \hookrightarrow \mathbb{I}^1$ iff it is equivalent to a set. $\triangle$

Example 6.9. A functor of 1-categories is orthogonal to $\Theta^1 \hookrightarrow \mathbb{I}^1$ iff it is conservative and iso-faithful, i.e. iso-reflecting and faithful on isomorphisms. $\triangle$

Example 6.10. A 2-category is orthogonal to $\Theta^2 \hookrightarrow \mathbb{I}^2$ iff it is biequivalent to a category. $\triangle$