Localizing what??

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Contents

In this note we study two kinds of localization and their intersection. While we focus on classical category theory, the ideas and theorems generalize to higher categories (see **??**).

1 Localizations

Recall the usual definition of localization of a category at a collection of morphisms.

Definition 1.1. A **localization** of a category $\mathcal{C}$ at a collection of morphisms W is a functor $L : \mathcal{C} \to \mathcal{C}[W^{-1}]$ sending W to isomorphisms in $\mathcal{C}[W^{-1}]$ satisfying the following universal property:

- If $F : \mathcal{C} \to \mathcal{D}$ is a functor sending W to isomorphisms in $\mathcal{D}$, then there exists a functor $\widetilde{F} : \mathcal{C}[W^{-1}] \to \mathcal{D}$ and a natural isomorphism $\sigma : F \cong \widetilde{F} \circ L$. Given another such pair $(\widetilde{F}', \sigma')$, there exists a unique natural isomorphism $\tau : \widetilde{F} \cong \widetilde{F}'$ such that $(\tau.L) \circ \sigma = \sigma'$.

Example 1.2. The localization of $\mathcal{C}$ at all morphisms is the groupoid obtained by inverting every morphism in $\mathcal{C}$. △

Notice that no conditions are imposed in W. The following alternative will prove itself useful.

Definition 1.3. A **localization** of a category $\mathcal{C}$ at a collection of morphisms W is a functor $L : \mathcal{C} \to \mathcal{C}[W^{-1}]$ such that

1. $\underline{よ}(L) := L^* : \mathbf{Cat}(\mathcal{C}[W^{-1}], \mathcal{E}) \to \mathbf{Cat}(\mathcal{C}, \mathcal{E})$ is fully faithful for every category $\mathcal{C}$.
2. the essential image of L^* consists of functors sending W to isomorphisms.

With this we can call a functor a localization without specifying W:

Definition 1.4. A functor $F : \mathcal{C} \to \mathcal{D}$ *exhibits* $\mathcal{D}$ *as a localization of* $\mathcal{C}$ if it is the localization of $\mathcal{C}$ at some collection of morphisms.

Yet, there is always a canonical characterization of W:

Proposition 1.5. *If* $F : \mathcal{C} \to \mathcal{D}$ *is a localization of* $\mathcal{C}$ *and* W *be the collection of morphisms* $f \in F$ *such that* Ff *is an isomorphism in* $\mathcal{D}$. *Then* $\mathcal{D} = \mathcal{C}[W^{-1}]$.

Proof. **[DT: to-do]** Suppose that F is a localization of $\mathcal{C}$ at W', so that $W' \subseteq W$ and hence $\mathcal{C}[W^{-1}] \to \mathcal{D}$. Then Definition 1.3 allows us to regard $\mathbf{Cat}(\mathcal{C}[W^{-1}], \mathcal{D})$ as a full subcategory of $\mathbf{Cat}(\mathcal{C}[W'^{-1}], \mathcal{E})$ ☐

Corollary 1.6. *A functor* $F : \mathcal{C} \to \mathcal{D}$ *is not a localization of* $\mathcal{C}$ *at* W *iff there exists a morphism* $f \in W$ *such that* Ff *is not an isomorphism.*

Saturation

Definition 1.7. A collection of morphisms $S \subseteq \mathcal{C}$ is **saturated** if there exists a functor $F : \mathcal{C} \to \mathcal{D}$ such that S is precisely the class of morphisms sent to isomorphisms by F.

Proposition 1.8. *If* $S \subseteq \mathcal{C}$ *is saturated, then it satisfies 2-out-of-3 and contains all isomorphisms.*

Proof. This is very easy. ☐

Proposition 1.9. *A collection of morphisms* $S \subseteq \mathcal{C}$ *is saturated iff* S *is precisely the class of morphisms inverted by* $\mathcal{C} \to \mathcal{C}[S^{-1}]$.

Proof. Assume that S is saturated via a functor $F : \mathcal{C} \to \mathcal{D}$. Notice that ☐

Definition 1.10. The **saturation** of a class of morphisms $S \subseteq \mathcal{C}$ is is precisely the collection $\bar{S}$ of morphisms sent to isomorphisms by $L : \mathcal{C} \to \mathcal{C}[S^{-1}]$.

Proposition 1.11. *The saturation of* $S \subseteq \mathcal{C}$ *is the smallest saturated class of morphisms containing* S.

Proof. ☐

Proposition 1.12. $\mathcal{C}[S^{-1}] \cong \mathcal{C}[\bar{S}^{-1}]$

Proof. The converse is obvious. ☐

2 Reflective subcategories

Definition 2.1. A **reflective subcategory** of a category $\mathcal{C}$ is a full subcategory $\mathcal{D}$ whose inclusion functor $i : \mathcal{D} \hookrightarrow \mathcal{C}$ has a left adjoint L.

Example 2.2. A category is *gaunt* if it has no non-trivial isomorphisms. The full subcategory inclusion **Gaunt** $\hookrightarrow$ **Cat** admits a left adjoint that "gauntifies" a category by first identifying isomorphic objects, then discarding the resulting automorphisms. △

Example 2.3. The full subcategory inclusion **Gpd** $\hookrightarrow$ **Cat** admits a left adjoint which sends a category to the groupoid obtained by inverting all morphisms. △

Example 2.4. A category is *contractible* or *(-2)-truncated* if it is equivalent to a point. The full subcategory inclusion $\mathbf{Cat}_{\leq -2} \hookrightarrow \mathbf{Cat}$ admits a left adjoint which sends a category to the codiscrete groupoid on its objects. △

Lemma 2.5. *Let $i : \mathcal{D} \hookrightarrow \mathcal{C}$ is a reflective subcategory with $L \dashv i$. Then*

1. *the counit $\varepsilon : Li \Rightarrow 1_{\mathcal{C}}$ is a natural isomorphism.*

2. *whiskering L with the unit $\eta : 1_{\mathcal{D}} \Rightarrow RL$ defines a natural isomorphism $L\eta : L \cong LRL$.*

Proof. The action of i on morphisms factors by pulling back with the counit:

$$i : \mathcal{C}(x, a) \xrightarrow{\varepsilon_x^*} \mathcal{C}(Lix, a) \cong \mathcal{C}(ix, ia).$$

The composite map is a bijection iff ε_x^* is a bijection iff ε_x is a natural isomorphism, proving (1).

For (2), applying the inverse of ε_{Lx} to the triangle equation $\varepsilon_{Lx} \circ L\eta_x = 1_{L_x}$ shows that $L\eta_x = \varepsilon_{Lx}^{-1}$, which is invertible. □

Proposition 2.6. *If $i : \mathcal{D} \hookrightarrow \mathcal{C}$ is a reflective subcategory, then the left adjoint $L \dashv i$ is a localization of $\mathcal{C}$ at $W := \{f \in \mathcal{C} : Lf$ is an isomorphism$\}$.*

Proof. Let $F : \mathcal{C} \to \mathcal{E}$ be a functor sending W to isomorphisms in $\mathcal{E}$, and define the functor $\widetilde{F} := F \circ i : \mathcal{D} \to \mathcal{E}$. Lemma 2.5 implies that the components of the unit are in W, so F takes them to isomorphisms in $\mathcal{E}$. It follows that $F\eta : F \cong \widetilde{F} \circ L$ is a natural isomorphism.

Next we show uniqueness up to unique natural isomorphism. If $(\hat{F}, \rho : \hat{F}L \cong F)$ is another extension of F via L, then pasting with ε defines a natural isomorphism $\sigma : \hat{F} \cong Fi =: \widetilde{F}$:

$$\begin{tikzcd}
& \mathcal{C} \arrow[rr, "F"] \arrow[dr, "L"] \arrow[d, phantom, "\varepsilon"] & & \mathcal{E} \\
\mathcal{D} \arrow[ur, "i"] \arrow[rr, equal] & {} & \mathcal{D} \arrow[ur, "\hat{F}"'] \arrow[u, phantom, "\rho"] &
\end{tikzcd}$$

The compatibility condition $(\sigma L)(F\eta) = \rho$ follows from one of the triangle identities for $L \dashv i$:

$$(\sigma L)(F\eta) = \begin{tikzcd}[column sep=small]
\mathcal{C} \arrow[rr, equal] \arrow[dr, "L"'] & {} \arrow[d, phantom, "\eta"] & \mathcal{C} \arrow[rr, "F"] \arrow[dr, "L"'] \arrow[d, phantom, "\varepsilon"] & {} \arrow[d, phantom, "\rho"] & \mathcal{E} \\
& \mathcal{D} \arrow[ur, "i"'] \arrow[rr, equal] & {} & \mathcal{D} \arrow[ur, "\hat{F}"'] &
\end{tikzcd} = \begin{tikzcd}[column sep=small]
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "L"'] & {} \arrow[d, phantom, "\rho"] & \mathcal{E} \\
& \mathcal{D} \arrow[ur, "\hat{F}"'] &
\end{tikzcd} = \rho.$$

The other triangle identity implies that any other compatible natural isomorphism $\sigma' : \hat{F} \cong \widetilde{F}$ is actually equal to σ:

$$\begin{tikzcd}[column sep=small]
& \mathcal{C} \arrow[rr, "F"] & {} \arrow[d, phantom, "\sigma'"] & \mathcal{E} \\
\mathcal{D} \arrow[ur, "i"] \arrow[rr, equal] & & \mathcal{D} \arrow[ur, "\hat{F}"'] &
\end{tikzcd} = \begin{tikzcd}[column sep=small]
& \mathcal{C} \arrow[rr, equal] \arrow[dr, "i"'] \arrow[d, phantom, "\varepsilon"] & {} \arrow[d, phantom, "\eta"] & \mathcal{C} \arrow[rr, "F"] & {} \arrow[d, phantom, "\sigma'"] & \mathcal{E} \\
\mathcal{D} \arrow[ur, "L"] \arrow[rr, equal] & {} & \mathcal{D} \arrow[ur, "i"'] \arrow[rr, equal] & & \mathcal{D} \arrow[ur, "\hat{F}"'] &
\end{tikzcd} = \begin{tikzcd}[column sep=small]
& \mathcal{C} \arrow[rr, "F"] \arrow[dr, "L"'] \arrow[d, phantom, "\varepsilon"] & {} \arrow[d, phantom, "\rho"] & \mathcal{E} \\
\mathcal{D} \arrow[ur, "i"] \arrow[rr, equal] & {} & \mathcal{D} \arrow[ur, "\hat{F}"'] &
\end{tikzcd} =: \sigma.$$

$\square$

Remark 2.7. A *reflective localization* is an adjoint pair whose right adjoint is fully faithful, or equivalently whose counit is invertible. Reflective subcategories are a particular case of reflective localizations, and most results in this section also hold for reflective localizations. In fact, if $L : \mathcal{C} \leftrightarrows \mathcal{D} : R$ is a reflective localization, then the essential image of R is a reflective subcategory of $\mathcal{C}$.

Remark 2.8. Reflective localizations can be regarded as a categorification of *idempotent splitting*. First notice that if $L \dashv i$ is a reflective localization then the reflection $T := iL$ is an idempotent monad as the canonical multiplication $iLiL \xrightarrow{i\varepsilon L} iL$ is an isomorphism $T^2 \cong T$. Notice that this definition makes sense in any 2-category. A weaker variant of idempotent splitting is used by Douglas-Reutter to define semisimple 2-categories.

Non-examples

The following non-example is only a reflective subcategory in the 2-categorical sense.

Example 2.9. A *flagged category* is an essentially surjective functor $F : \mathcal{G} \to \mathcal{C}$, where $\mathcal{G}$ is a groupoid. An ordinary category $\mathcal{C}$ has a canonical flagging given by $\mathbf{ob}(\mathcal{C}) \hookrightarrow \mathcal{C}$, and this construction determines a fully faithful functor $\mathbf{Cat} \to \mathbf{Cat}_{\text{flagged}}$. This functor is in fact a right adjoint, and the reflective localization $L : \mathbf{Cat}_{\text{flagged}} \to \mathbf{Cat}$ sends $F : \mathcal{G} \to \mathcal{C}$ to the quotient category $\mathcal{C}/\sim$ defined by the congruence relation generated by $Ff \sim 1_{s(f)}$. $\triangle$

Question 2.10. *Are univalent categories S-local with respect to a generating set S? (c.f. Corollary 4.8) In other words, is the a small collection of functors of flagged categories S such that a category is univalent iff it is orthogonal to S?*

The following non-example is only a reflective subcategory in the ∞-sense.

Example 2.11. A space is *n-truncated* if its homotopy groups vanish above degree n. The inclusion $\mathcal{S}_{\leq n} \hookrightarrow \mathcal{S}$ admits a left adjoint which sends a space to its truncation. $\triangle$

Remark 2.12. The full subcateogry n-connected spaces is a *co*reflective subcategory, as the inclusion $\mathcal{S}_{\geq n} \hookrightarrow \mathcal{S}$ admits a *right* adjoints.

3 Localization at local objects

In this section S is a collection of morphisms of a category $\mathcal{C}$.

Definition 3.1. An object $c \in \mathcal{C}$ is **S-local** if $f^* : \mathcal{C}(b, c) \to \mathcal{C}(a, c)$ is a bijection for every $f : a \to b$ in S.

Remark 3.2. Being S-local means that extension problems against S have unique solutions:

$$\begin{tikzcd} x \arrow[r] \arrow[d, "S\ni"'] & c \\ y \arrow[ur, dashed, "\exists!"] & \end{tikzcd}$$

Example 3.3. Let J denote the walking isomorphism. The local objects of **Cat** with respect to the terminal map $\exists! : J \to *$ are precisely the gaunt categories. △

Example 3.4. Let I denote the walking morphism. The local objects of **Cat** with respect to one of the non-trivial inclusions $I \hookrightarrow J$ are precisely the groupoids. △

Example 3.5. A category is contractible iff it is local with respect to the morphism $\partial I \hookrightarrow I$. △

Notation 3.6. A full subcategory inclusion $i : \mathcal{D} \hookrightarrow \mathcal{C}$ induces a restricted Yoneda embedding $よ_{\mathcal{D}} : \mathcal{C} \to \mathbf{Set}^{\mathcal{D}^{\mathrm{op}}}$ given by $c \mapsto \mathcal{C}(c, i(-))$. For the remainder of this section i denotes the full subcategory inclusion $i : \mathcal{C}_S \hookrightarrow \mathcal{C}$ of the S-local objects.

Definition 3.7. A morphism $f : x \to y$ is **S-local** if $よ_{\mathcal{D}}(f)$ is an isomorphism. In other words, $f^* : \mathcal{C}(y, c) \to \mathcal{C}(x, c)$ is an isomorphism for every S-local object c.

Remark 3.8. The S-local equivalences is, by the definition, the saturation of S.

Lemma 3.9. *The S-local morphisms always satisfy 2-out-of-3.*

Proof. This is true for any saturated class of morphisms (Proposition 1.8). □

Lemma 3.10. *Suppose that S satisfies 2-out-of-3 and contains identities. If c and d are S-local objects then $f : c \to d$ is an S-local equivalence iff it is an isomorphism.*

Proof. The following lift provides a left inverse to f:

$$\begin{tikzcd} c \arrow[r, equal] \arrow[d, "f"'] & c \\ d \arrow[ur, dashed, "\exists!\, g"'] & \end{tikzcd}$$

Then g is still in S by 2-out-of-3. By the same argument it also has a left inverse, which must be equal to f since left and right inverses must agree. □

Proposition 3.11. *If S-local objects form a reflective subcategory $i : \mathcal{C}_S \hookrightarrow \mathcal{C}$, then the left adjoint $L \dashv i$ is a localization of $\mathcal{C}$ at the S-local morphisms.*

Proof. By Proposition 2.6, it suffices to show that the S-local morphisms are precisely those morphisms inverted by L. Indeed, a morphism $f : x \to y$ is S-local iff $よ_{\mathcal{C}_S}(f) = f^* : \mathcal{C}(y, i(c)) \to \mathcal{C}(x, i(c))$ is a bijection for every $c \in \mathcal{C}_S$. Tranposing the last equation we obtain the equivalent condition that $Lf^* : \mathcal{C}_S(Ly, c) \to \mathcal{C}_S(Lx, i(c))$ is a bijection for every $c \in \mathcal{C}_S$, but this is precisely the Yoneda embedding applied to $Lf \in \mathcal{C}_S$, so it holds iff Lf is an isomorphism. □

Proposition 3.12. *If $i : \mathcal{D} \hookrightarrow \mathcal{C}$ is a reflective subcategory with $\mathcal{D} = \mathcal{C}[S^{-1}]$, then the essential image of i is precisely the full subcategory of S-local objects.*

Proof. This follows from Remark 2.7. □

A common situation is that we have shown that $L : \mathcal{C} \to \mathcal{D} : i$ is a reflective localization which we want to characterize as a reflection at some S-local objects i.e. we want to determine the essential image of i.

Question 3.13. *Let S be a class of morphisms such that $\mathcal{C}_S \hookrightarrow \mathcal{C}$ is a reflective subcategory with left adjoint L. Then $\mathcal{C}_S = \mathcal{C}[W^{-1}]$ where W is the collection of morphisms that L sends to isomorphisms. Definitely $S \subseteq W$, so that there is a functor $\mathcal{C}[S^{-1}] \to \mathcal{C}[W^{-1}] \cong \mathcal{C}_S$. When is this an equivalence?*

4 Locally presentable categories

Proposition 4.1. *Let $\mathcal{C}$ be a locally presentable category and $\mathcal{D} \hookrightarrow \mathcal{C}$ a full subcategory. Then the following are equivalent:*

1. *$\mathcal{D}$ is the subcategory of S-local objects for set of morphisms.*
2. *$\mathcal{D}$ is a reflective subcategory closed under filtered colimits.*

Proof. This is Theorem 1.39 in Adámek-Rosický. □

Proposition 4.2. *Let $\mathcal{C}$ be a category. The following are equivalent:*

1. *$\mathcal{C}$ is locally presentable.*
2. *$\mathcal{C}$ is the category of continuous presheaves on some category $\mathcal{A}$.*
3. *$\mathcal{C}$ is a reflective category of $\mathbf{Set}^{\mathcal{A}}$ closed under filtered colimits for some category $\mathcal{A}$.*
4. *$\mathcal{C}$ is the category of S-local objects of $\mathbf{Set}^{\mathcal{A}}$ for some category $\mathcal{A}$.*
5. *$\mathcal{C}$ is the completion of a category $\mathcal{A}$ under filtered colimits.*

Proof. This is Theorem 1.46 in Adámek-Rosický. □

[DT: something something sheaves]

Lemma 4.3. *Let S be a collection of morphisms in a category $\mathcal{C}$ and consider the full subcategory $\mathcal{C}_S \hookrightarrow \mathcal{C}$ on the S-local objects. Then $\mathcal{C}_S$ is closed under limits. If the domain and codomain of morphisms in S are compact, then $\mathcal{C}_S$ is also closed under filtered colimits.*

Proof. Use the hypotheses to take the (co)limits out of the hom-sets and apply the definition of S-local objects. ☐

Corollary 4.4. *Let $\mathcal{C}$ be a locally presentable category and let S be a collection of morphisms whose domains and codomains are compact. Then $\mathcal{C}_S \hookrightarrow \mathcal{C}$ is a reflective subcategory exhibiting $\mathcal{C}[S^{-1}]$.*

Proof. Combine Lemma 4.3 with Proposition 4.1. ☐

Remark 4.5. The definitions and theorems of reflective localizations and locally presentable categories parse verbatim to ∞-categories. If a specific model desired, then the *Bousfield localization* at a space of morphisms presents the category of S-local objects. This construction always exits when the model structures are *combinatorial*, i.e. when the ∞-categories are presentable.

Codiagonal completion

Let $\mathcal{C}$ be a locally presentable category and $\mathcal{D}$ be a reflective subcategory, so that $\mathcal{D} \cong \mathcal{C}[W^{-1}]$ for a saturated class of morphisms W (combine Remark 3.8 with Proposition 3.11). What does it mean to say that W is generated by a set I?

Then for any set of morphisms $I \subseteq \mathcal{C}$ we can produce a *weak* factorization systems via the small object argument. Namely, the factorization system is $(\mathrm{cell}(I), \mathrm{rlp}(I))$, where $\mathrm{cell}(I)$ can equivalently regarded as llp(rlp(I)) or the collection of I-cell complexes, i.e. the closure of I under pushouts, transfinite composition, and retracts (in the arrow category $\mathbf{Arr}(\mathcal{C})$).

On the other hand, we can turn $\mathrm{rlp}(I)$ into $\mathrm{orlp}(I)$ (unique lifts) by adding codiagonals, that is, given any $f : x \to y$ in I also add the morphism $y \sqcup_x y \to y$ induced by the identities on y:

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[d, "f"'] & y \arrow[d] \arrow[ddr, bend left, equal] & \\
y \arrow[r] \arrow[drr, bend right, equal] & y \sqcup_x y \arrow[dr, dashed, "\delta_f"] & \\
& & b
\end{tikzcd}$$

Then a morphism has unique lifts against I iff it has lifts against $\widetilde{I}$.

Now run the small object argument to obtain the weak factorization system $(\mathrm{cell}(\widetilde{I}), \mathrm{rlp}(\widetilde{I}))$, which is in fact an orthogonal factorization system.

Suppose that $\mathcal{C}$ has terminal objects. Then, by definition,

$$x \in \mathcal{C} \text{ is } I\text{-local} \leftrightsquigarrow x \xrightarrow{\exists!} * \in \mathrm{rlp}(\widetilde{I}).$$

Likewise a morphisms is an I-local equivalence iff it has the LLP against I-local objects. Hence the I-local equivalences are *precisely* the $\widetilde{(I)}$-cell complexes. By virtue of Remark 3.8, we have shown the following.

Corollary 4.6. *If I is a class of morphisms in a locally presentable category, then the following sets are the same:*

1. *the I-local equivalences.*

2. *the $\widetilde{I}$-cell complexes, where $\widetilde{I}$ is the codiagonal completion of I.*

3. *the saturation of I.*

Now suppose that you show that a subcategory $\mathcal{D} \hookrightarrow \mathcal{C}$ is reflective, and moreover that the objects of $\mathcal{D}$ are I-local morphisms with respect to a small class morphisms. Then you can describe the S-local morphisms sharply: they are $\widetilde{I}$-cell complexes.

Corollary 4.7. *Let $\mathcal{C}$ be a locally presentable category and $\mathcal{D} \hookrightarrow \mathcal{C}$ be a reflective category of S-local objects for a set of morphisms $S \subseteq \mathcal{C}$. Suppose that the full subcategory* $\mathbf{Arr}(\mathcal{C})$ *on S is accessibly embedded, i.e. closed under filtered colimits. Then $\mathcal{D} = \mathcal{C}[S^{-1}]$.*

Proof. **[DT: to-do, the point is that the additional condition should correspond to $\mathcal{D}$ being closed under filtered colimits, as suggested by the first theorem in this section]**

□

The following corollary is rephrasing Corollary 4.4 in the language of this subsection.

Corollary 4.8. *Let $\mathcal{C}$ be a locally presentable category and suppose that W is a saturated set of morphisms of the form $W = cell(I)$ for a set of morphisms I. Then the full subcategory on I-local objects presents $\mathcal{C}[W^{-1}] \cong \mathcal{C}[I{-}1]$.*

Example 4.9. Let $\mathcal{C} = \mathbf{Cat}$ and $I = \{\partial\Theta_1 \hookrightarrow \Theta_1\}$ (see Construction 5.4), so that a functor is orthogonal to I iff it is fully faithful. The codiagonal completion $\widetilde{I}$ of I adds the morphism $\partial\Theta_2 \twoheadrightarrow \Theta_1$, hence there is an orthogonal factorization system $(\mathrm{cell}(\widetilde{I}), \mathrm{orlp}(I))$. This must be equal to the factorization system (eso, f.f.), hence $\mathrm{cell}(\widetilde{I})$ is precisely the collection of fully faithful functors.

Moreover, a category is I-local iff it is a contractible groupoid, but we have seen in Example 3.5 that these form a reflective subcategory of **Cat**. Hence $\mathbf{Cat}_{\leq -2}$ is precisely the localization of **Cat** at $\partial\Theta_1 \hookrightarrow \Theta_1$. △

5 ω-categories

[DT: In progress for the ATCAT.]

Definition 5.1. A category is a *fixed point for enrichment* if $\mathcal{V} \cong \mathcal{V}\mathbf{Cat}$.

Proposition 5.2. *The category* $\mathbf{Cat}_\omega$ *of* ω*-categories is a terminal object for the full subcategory of* **SymMonCat** *at the fixed points for enrichment.*

Proof. This is due to Goldthorpe. □

Definition 5.3. The *suspension* of an ω-category $\mathcal{C}$ is the ω-category with two objects and whose unique hom-ω-category is $\mathcal{C}$.

Construction 5.4. The n*-globe* is the n-category Θ_n defines as the n-th iterated suspension of the point $*$:

$$\Theta_0 = \ * \ , \qquad \Theta_1 = \ * \longrightarrow * \qquad \Theta_2 = \begin{tikzcd} * \arrow[r, bend left, ""{name=U}] \arrow[r, bend right, ""'{name=D}] & * \arrow[Rightarrow, from=U, to=D] \end{tikzcd} \qquad \ldots$$

The *boundary of the* n*-globe* is the $(n-1)$-category $\partial\Theta_n$ obtained by discarding the top morphism:

$$\partial\Theta_0 = \ \emptyset \ , \qquad \partial\Theta_1 = \ * \qquad * \qquad \partial\Theta_2 = \begin{tikzcd} * \arrow[r, bend left] \arrow[r, bend right] & * \end{tikzcd} \qquad \ldots$$

Proposition 5.5. *Let* $S_{\geq n}$ *denote the set of boundary inclusions* $\partial\Theta_i \hookrightarrow \Theta_i$ *for* $i \geq n$*. Then an* ω*-category is* $S_{\geq n}$*-local iff it is an* n*-category.*

Corollary 5.6. *The subcategory of* n*-categories forms a reflective subcategory of* $\mathbf{Cat}_\omega$*.*

Proof. **[DT: to-do]** Since $\mathbf{Cat}_\omega$ is locally presentable there is aS there is a reflection onto the subcategory of S-local objects. □

6 Localizing at a prime p

[DT: Just a sketch. The goal is to pinpoint results analogous to the yoga of p-local spectra.]

Let R be a PID and $R_\mathfrak{p}$ the localization of R *away* from a prime ideal $\mathfrak{p} \subset R$.

$$R_\mathfrak{p} = \left\{ \frac{r}{s} : r \in R, s \in R \setminus \mathfrak{p} \right\}$$

Example 6.1. The ring $\mathbb{Z}_{(2)}$ consists of rational numbers with odd denominator. △

Proposition 6.2. *The functor* $\mathbf{Mod}_R \to \mathbf{Mod}_{R_\mathfrak{p}}$ *defined by* $M \mapsto M \otimes_R R_\mathfrak{p}$ *is a reflective localization.*

Proof. **[DT: to-do]**

□

Proposition 3.12 implies that $\mathbf{Mod}_{R_\mathfrak{p}}$ is equivalent to the category of S-local modules for some class of homomorphisms S. We will give a sharper description of this subcategory.

Proposition 6.3. *An R-module M is S-local iff the endomorphism defined by $m \mapsto r \cdot m$ is invertible for $r \notin \mathfrak{p}$ iff it is local with respect to the canonical homomorphism $R \to R_{\mathfrak{p}}$.*

Proof. □

Corollary 6.4. *A finitely generated R-module is $\mathfrak{p}$-local iff it only has $\mathfrak{p}$-torsion, i.e. the endomorphism defined by $m \mapsto r \cdot m$ is nilpotent iff $r \in \mathfrak{p}$.*

Example 6.5. A finite abelian group is p-local iff it only has no p-torsion. △

Proposition 6.6. *A homomorphism of R-modules is $\mathfrak{p}$-local iff it induces isomorphisms on the $\mathfrak{p}$-torsion subgroups.*