On the uniqueness of high school angles

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Definition 1. A **high school angle** is a first quadrant angle that is a rational multiple of π, including 0 and $\pi/2$, and whose cosine is of the form $\sqrt{k}/n$ for integers k and n.

Lemma 2. *Let ξ be algebraic over $\mathbb{Q}$. Then $[\mathbb{Q}(\xi) : \mathbb{Q}(\xi + \xi^{-1})] \leq 2$.*

Proof. This follows from the fact that ξ is a root of the quadratic polynomial $x^2 - (\xi + \xi^{-1}) \cdot x + 1$ with coefficients in $\mathbb{Q}(\xi + \xi^{-1})$. □

Lemma 3. *Let ξ be a root of unity of order n. Then $[\mathbb{Q}(\xi) : \mathbb{Q}] = \varphi(n)$, where φ is the Euler totient function.*

Proof. The minimal polynomial of an n-th root of unity is the n-th cyclotomic polynomial, whose degree is $\varphi(n)$. Hence the degree of the extension $[\mathbb{Q}(\xi) : \mathbb{Q}]$ is $\varphi(n)$. □

Lemma 4. *Let ζ be the cosine of a nonzero angle of the form $\theta = p\pi/q$, where p and q are coprime. Then $[\mathbb{Q}(\zeta) : \mathbb{Q}]$ is either $\varphi(q)/2$ or $\varphi(q)$.*

Proof. Notice $\zeta \in \mathbb{Q}(\xi + \xi^{-1})$, where $\xi = e^{i\theta}$; moreover n is the order of ξ. Hence $[\mathbb{Q}(\zeta) : \mathbb{Q}] = [\mathbb{Q}(\xi + \xi^{-1}) : \mathbb{Q}]$, so Lemma 3 gives us $[\mathbb{Q}(\zeta) : \mathbb{Q}] = \varphi(q)$. This, combined with Lemma 2 and the degree equation

$$[\mathbb{Q}(\xi) : \mathbb{Q}(\xi + \xi^{-1})] \cdot [\mathbb{Q}(\xi + \xi^{-1}) : \mathbb{Q}] = [\mathbb{Q}(\xi) : \mathbb{Q}]$$

implies the result. □

Theorem 5. *The only high school angles are 0, $\pi/6$, $\pi/4$, $\pi/3$, and $\pi/2$.*

Proof. Let $\theta = 2p\pi/q$ be a high school angle and $\zeta := \cos\theta$. Note that $[\mathbb{Q}(\zeta) : \mathbb{Q}] \leq 2$, as ζ lies in a quadratic extension of $\mathbb{Q}$. On the other hand, Lemma 4 informs us that $[\mathbb{Q}(\zeta) : \mathbb{Q}] \leq \varphi(q)/2$. So $\varphi(q) \leq 4$, whence the possible values for q are 1, 2, 3, 4, 5, 6, 8, 10, and 12. The first quadrant condition, $0 < 2p \leq q$, excludes the numbers 1 through 3. The pairs $(p, q) \in \{(1, 5), (1, 10), (3, 10)\}$ are excluded by direct inspection. The remaining options are those in the statement of the theorem. □

Remark 6. For a given n, list the numbers 0 through n^2, put square roots in each number, and divide each one by n. The *high school dream* is that this list comes from a list of high school numbers. We have just proved that $n = 2$ is the only non-trivial high school dream via finding all high school angles, but we had to directly inspect the cosine of $\pi/10$, $3\pi/10$, and $\pi/5$. However, we could have directly derived this weaker result from observing that $2\cos\theta$ is an algebraic integer.