Some results for adjunctable 2-categories

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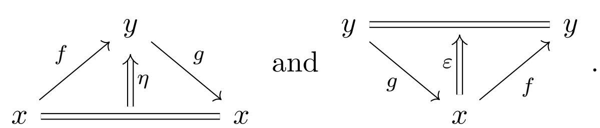
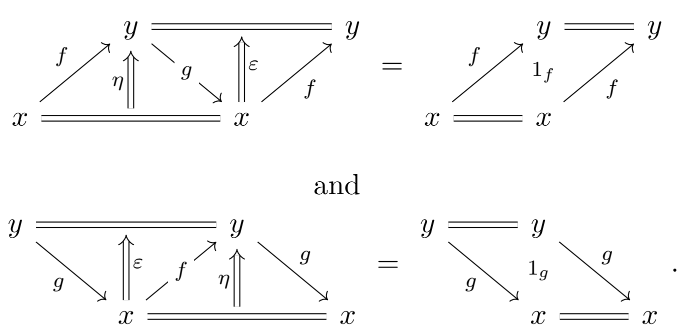
It seems that the category of theory of (∞,∞)-categories with adjoints behave like the homotopy theory of spaces, except that we need to keep track of the 1-categories $\Pi_n(\mathcal{C}) := h_1(\Omega^n\mathcal{C})$ instead of the sets $\pi_n(X) := h_0(\Omega^n X)$. In this note we give one example of such phenomena. There is no claim to originality: we are adapting the last theorem in [H25].

Definition 1. A morphism $f : x \to y$ is a **left adjoint** if there exists a morphism $g : y \to x$ and a pair of 2-morphisms



such that the *triangle identities* hold:



Example 2. An object x in a monoidal category $(\mathcal{M}, \otimes)$ is a left dual precisely if it is a left adjoint when regarded as a morphism in $\mathcal{M}$. △

Proposition 3. *A morphism $f : x \to y$ in $\mathcal{C}$ is a left adjoint if and only if the induced functor $f_* : \mathcal{C}(z,x) \to \mathcal{C}(z,y)$ is a left adjoint functor for each $z \in \mathcal{C}$, naturally in z.*

Sketch of proof. If $f \dashv f^R$, then there is an induced adjunction $f_* : \mathcal{C}(z,x) \leftrightarrows \mathcal{C}(z,y) : f^R_*$, with unit and counit given by whiskering η and ε. Conversely, if $f_* \dashv G$, then the adjunction $f \dashv f^R$ is obtained by setting $f^R := G(1_y)$. The unit is obtained by transposing 1_f along $f_* \dashv G$:

$$1_f \in \mathcal{C}(x,y)(f,f) \simeq \eta \in \mathcal{C}(x,x)(1_x, f^R f)$$

The counit is obtained by transposing 1_{f^R} to $\varepsilon : ff^R \to 1_y$, and the triangle identities follow from the triangle identities for $f_* \dashv G$. □

Definition 4. A 2-functor $F : \mathcal{C} \to \mathcal{D}$ is a **2-equivalence** if the induced functors

$$h_1F : h_1\mathcal{C} \to h_1\mathcal{D} \quad \text{and} \quad F_{x,y} : \mathcal{C}(x,y) \to \mathcal{D}(Fx, Fy)$$

are equivalences of categories for every $x, y \in \mathcal{C}$.

The following result shows that the condition on $F_{x,y}$ only needs to be checked on endomorphisms, provided that $\mathcal{C}$ admits left adjoints.

Theorem 5 (Heine)**.** *Let $\mathcal{C}$ be a 2-category that admits left adjoints. A 2-functor $F : \mathcal{C} \to \mathcal{D}$ is a 2-equivalence if and only if the induced functors*

$$h_1F : h_1\mathcal{C} \to h_1\mathcal{D} \quad \textit{and} \quad F_{x,x} : \mathcal{C}(x,x) \to \mathcal{D}(Fx, Fx)$$

are equivalences of categories for every $x \in \mathcal{C}$.

Proof. The forward direction is clear from Definition 4, so we show the converse.

Essential surjectivity for each $F_{x,y}$ follows from the equivalence h_1F, so it remains to show that $F_{x,y}$ is fully faithful. In other words, that $F_{x,y} : \mathcal{C}(x,y)(f,g) \to \mathcal{D}(Fx,Fy)(Ff,Fg)$ is a bijection for every $f, g : x \to y$.

Pick a right adjoint $f \dashv f^R$. The induced adjunction $f_* : \mathcal{C}(x,y) \leftrightarrows \mathcal{C}(x,y) : f^R_*$ yields an equivalence

$$f_* : \mathcal{C}(x,y)(f,g) \simeq \mathcal{C}(x,x)(1_x, f^R f) : f^R_*.$$

Adjunctions are preserved by F, so $Ff \dashv Ff^R$, and hence the equivalence

$$Ff_* : \mathcal{D}(Fx,Fy)(Ff,Fg) \simeq \mathcal{D}(Fx,Fx)(1_{Fx}, Ff^R Ff) : Ff^R_*.$$

Under these identifications, the map $F_{x,y}$ identifies with

$$\mathcal{C}(x,y)(f,g) \simeq \mathcal{C}(x,x)(1_x, f^R f) \xrightarrow{F_{x,x}} \mathcal{D}(Fx,Fx)(1_{Fx}, Ff^R Ff) \simeq \mathcal{D}(Fx,Fy)(Ff,Fg),$$

which is a bijection since $F_{x,x}$ is an equivalence by hypothesis. □

Corollary 6. *In the situation of Theorem 5, if both categories are connected, then it suffices to check that $F_{x,x}$ is an equivalence for a single $x \in \mathcal{C}$.*

References

[H25] Heine, H. (2025). *On the Categorification of Homology*. Preprint. Available at: https://arxiv.org/abs/2505.22640v1