Presentability

"Presentable categories are big enough to contain real math, but small enough to be manageable" —ChatGPT

- Classical def$^{\underline{n}}$: A 1-category $\mathcal{C}$ is presentable if
  1. $\mathcal{C}$ is locally small
  2. $\mathcal{C}$ is co-complete
  3. $\exists$ a set $S \hookrightarrow \mathrm{Obj}(\mathcal{C})$ of λ-compact objects that generates $\mathcal{C}$ under λ-filtered colimits, where λ is some regular cardinal.

Sub-def$^{\underline{ns}}$:

$X \in \mathcal{C}$ is λ-compact if: $\mathcal{C}(X,-): \mathcal{C} \longrightarrow \mathrm{Set}$ preserves λ-filtered colimit.

By "diagram", I mean the "shape category" whereas a commuting diagram is a "diagram in $\mathcal{C}$" i.e. $F: \text{diagram} \to \mathcal{C}$

A diagram, is λ-filtered if, $\forall$ sub-diagram of size $< \lambda$ ↳ any sub-cat of diagram

A λ-filtered colimit in $\mathcal{C}$ is then a colimit of a λ-filtered diagram in $\mathcal{C}$.

A regular cardinal λ means that any set of size λ cannot be constructed from $< \lambda$-many unions of smaller sets

↳ e.g. $|\mathbb{N}|$ b/c $\mathbb{N} \neq \bigcup_{i \in I} X_i$ $\forall\, |I|, |X_i| < |\mathbb{N}|$, similarly $|\mathbb{R}|$, etc.

e.g. $\bullet \rightrightarrows \bullet$ is not filtered, but $\bullet \rightrightarrows \bullet \to \bullet$ is

Kinda like sub-diagrams have meets or joins?

Examples:

1. Set; is locally small, has coproducts & coequalizers.

   Let $\lambda = |\mathbb{N}| = \omega$. $\forall X \in \mathrm{Set}$, $X = \operatorname{colim}_{F \subseteq X,\, |F| < \infty} F$

   Filtered colimits in Set are of the form $S = \bigcup_{i \in I} S_i$ s.t. $\forall i, j, \exists k$ s.t. $S_i \cup S_j \subseteq S_k$

ω-compact sets are the finite sets; $|X|<\infty \Rightarrow$ the functor $\mathrm{Set}(X,-):\mathrm{Set}\to\mathrm{Set}$ preserves ω-filtered limits ¯_(ツ)_/¯

↳ Moreover, the collection of all finite sets is a (small) set.

2. Grp is locally small, has free products & quotients. Again, $\lambda=\omega$ and ω-compact groups are finitely-presentable groups. Collection of fin. pres. grps is a set.

$\forall G\in \mathrm{Grp};\ G = \operatorname*{colim}_{H \subseteq_{\substack{fin\\gen}} G} H$

3. FinSet & FinGrp are <u>not</u> presentable b/c they are not co-complete.

4. Banach spaces & contractions are presentable w/ $\lambda = |\mathbb{R}|$

↳ Co-completeness is ~~required~~ useful, probably so that there is guaranteed to exist some λ. Dropping co-completeness gives the concept of "accessible categories"

- Representation Theorem: λ a regular cardinal & $\mathcal{C}$ a cat, TFAE

1. $\mathcal{C}$ is λ-presentable
2. $\mathcal{C}$ is equivalent to a full reflective sub-category of $\mathrm{Cat}(A,\mathrm{Set})$ closed under λ-filtered colimits, for some small category A.

This also means "$\mathcal{C}$ is an accessibly embedded full reflective sub-cat of (co)pre-sheaves on a small cat A". (☆)

We'll use point ~~2~~ (☆) as an alternative def^{n} of presentability.

- Density Theorem: $F : \mathcal{C}^{op} \longrightarrow Set$ and $Elt(F)$ its category of elements, and $Forget : Elt(F) \longrightarrow \mathcal{C}$ the forgetful functor.

$$F = colim\left(Elt(F) \xrightarrow{Forget} \mathcal{C} \xhookrightarrow{\text{よ}} Cat(\mathcal{C}^{op}, Set) \right).$$

proof of representation thm: $\mathcal{C}$ λ-presentable. S the set of λ-compact generators of $\mathcal{C}$. $A := full(Subcat(S \subseteq Obj(\mathcal{C})))$

$$\text{よ}_A : \mathcal{C} \longrightarrow Cat(A^{op}, Set)$$
$$X \longmapsto \mathcal{C}(-, X)\big|_A$$

claim: よ_A is fully-faithful $\rightsquigarrow$ def^{n}: $A \hookrightarrow \mathcal{C}$ is "dense"

Hence, $\mathcal{C}$ is a full sub-cat of $Cat(A^{op}, Set)$, w/ A small.

Adjoint functor thm: $\mathcal{C}$ co-complete & よ_A preserves limits, or colimits?

so $\exists$ $L \dashv \text{よ}_A$ thus $\mathcal{C} \hookrightarrow Cat(A^{op}, Set)$ is reflective.

WTS: $\forall$ λ-filtered diagram in $\mathcal{C}$; $\text{よ}_A(\operatorname{colim}_{i \in I} c_i) \cong \operatorname{colim}_{i \in I} \text{よ}_A(c_i)$

$\forall a \in A$; $\text{よ}_A(\operatorname{colim}_{i \in I} c_i)(a) = \mathcal{C}(a, \operatorname{colim}_{i \in I} c_i) \cong \operatorname{colim}_{i \in I} \mathcal{C}(a, c_i)$ ▨