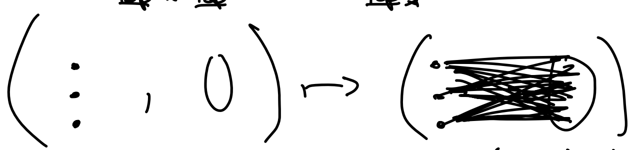
Ref: [§3.9–3.10, GHP]

Main idea: replace the Gray tensor product with the join (from 3.7–3.8).

Recall: 'Joins'

For topological spaces: $\mathrm{Top} \times \mathrm{Top} \xrightarrow{\vee} \mathrm{Top}_*$



$X \amalg_{\pi_0} (X \times Y \times I) \amalg_{\pi_1} Y$

For categories: $\mathrm{CAT} \times \mathrm{CAT} \xrightarrow{\vee} \mathrm{CAT}$

$(\mathcal{C}, \mathcal{D}) \longmapsto \mathcal{C} \vee \mathcal{D}$

obj: $\mathcal{C} \amalg \mathcal{D}$

mor:

$$\mathcal{C} \vee \mathcal{D}(A,B) = \begin{cases} \mathcal{C}(A,B) & A, B \in \mathcal{C}_0 \\ \mathcal{D}(A,B) & A, B \in \mathcal{D}_0 \\ \mathbb{1} & A \in \mathcal{C}_0, B \in \mathcal{D}_0 \\ \emptyset & \text{else} \end{cases}$$

eg) $\mathbb{1} \vee \mathbb{2} = (\bullet) \vee (\bullet \longrightarrow \bullet)$

$$\cong \begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[ur] \arrow[r] \arrow[r, phantom, shift left=3, "="] & \bullet \end{tikzcd}$$

For simplicial sets: $\Delta \times \Delta \xrightarrow{\oplus} \Delta$

$[n],[m] \longmapsto [n+m+1]$

restricts to a monoidal structure on $\Delta_+ = \Delta \cup \{-1\}$ (where $-1 \cong \bot$ $\cong \emptyset$)

with unit $\emptyset \in \Delta_+$ in degree -1.

(all acceptable notation. $\deg(-) = -1$)

eg) $[0] \oplus [1] = [2] \rightsquigarrow$

$[1] \oplus [1] = [3] \rightsquigarrow$

(a 3-simplex)

This extends to augmented simplicial sets $X, Y : \Delta_+^{op} \longrightarrow \underline{Set}$

by Day-conv: $X \otimes_{Day} Y =: X \star Y$

$$\begin{tikzcd}
(\Delta_+ \times \Delta_+)^{op} \arrow[r, "X \bar{\otimes} Y"] \arrow[d, "\oplus"'] \arrow[dr, phantom, "\Downarrow \eta"] & \underline{Set} \\
\Delta_+^{op} \arrow[ur, "X \star Y"'] &
\end{tikzcd}$$

where $(X \bar{\otimes} Y)_{n,m} = X_n \times Y_m$

$$X \star Y \cong \mathrm{lan}_{\oplus} X \bar{\otimes} Y \cong \underbrace{\int^{n,m} \Delta_+(n \oplus m, -) \times X_n \times Y_m}_{\text{Useful for calculations...}}$$

* Fubini's thm
* "Ninja" Yoneda
* Density ...

<u>Def</u>: The join of two simplicial sets $X, Y : \Delta^{op} \longrightarrow \underline{Set}$ is the join of their <u>trivial augmentations</u>.

$$\boxed{X_\emptyset = * = Y_\emptyset}$$

<u>LEMMAS/FACTS</u>

- $X \star Y$ is cocontinuous in each variable separately.
- $(\hat{\Delta}_+, \star)$ is closed monoidal with $[X, Y]_n = \hat{\Delta}_+(X, \mathrm{Dec}^{n+1} Y)$

$$\mathrm{Dec}^{n+1} Y = \mathrm{Dec}(Y)(-, n+1)$$

Total décalage functor.

This is precomposition with the augmented ordinal sum!

$$\left[\begin{tikzcd}
(\Delta_+ \times \Delta_+)^{op} \arrow[r, "X \times Y"] \arrow[d, "\oplus"'] \arrow[dr, phantom, "\Downarrow"] & \underline{Set} \\
\Delta_+^{op} \arrow[ur, dashed] & X \star Y
\end{tikzcd}\right] \rightsquigarrow \left[\begin{tikzcd}
\hat{\Delta}_+ \arrow[r, "\oplus^*"] \arrow[dr, "\mathrm{Dec}"'] & \widehat{\Delta_+ \times \Delta_+} \arrow[d, "\cong"] \\
& \hat{\hat{\Delta}}_+
\end{tikzcd}\right]$$

<u>Cat</u> is c.c.

The closure is formed: $\left(\underbrace{- \star - = \mathrm{lan}_{\oplus}(- \times -) \dashv \oplus^*}_{\text{by def'n.}} \right)$

- If $(\mathcal{C}, \otimes, I)$ is a $\mathcal{V}$-cat, for $\mathcal{V}$ a SMC, then the Yoneda embedding

$$(\mathcal{C}, \otimes, I) \xrightarrow{\;よ\;} (\hat{\mathcal{C}}, \otimes_{\text{Day}}, よ_I)$$

is strongly monoidal.

Pf: By coend calculus,

$$\begin{aligned}
(よ_X \otimes_{\text{Day}} よ_Y)(C) &\cong \int^{A,B} \mathcal{C}(C, A \otimes_{\mathcal{C}} B) \otimes_{\mathcal{V}} よ_X(A) \otimes_{\mathcal{V}} よ_Y(B) \\
\text{(Fubini)} \quad &\cong \int^{A,B} よ_{A \otimes_{\mathcal{C}} B}(C) \otimes_{\mathcal{V}} よ_X(A) \otimes_{\mathcal{V}} よ_Y(B) \\
&\cong \int^{A} \int^{B} \underbrace{よ_{A \otimes_{\mathcal{C}} B}(C) \otimes_{\mathcal{V}} よ_X(A)}_{G(B)} \otimes_{\mathcal{V}} よ_Y(B) \\
\text{(co-Yoneda)} \quad &\cong \int^{A} よ_{A \otimes_{\mathcal{C}} Y}(C) \otimes_{\mathcal{V}} よ_X(A) \\
&\cong よ_{X \otimes_{\mathcal{C}} Y}(C)
\end{aligned}$$

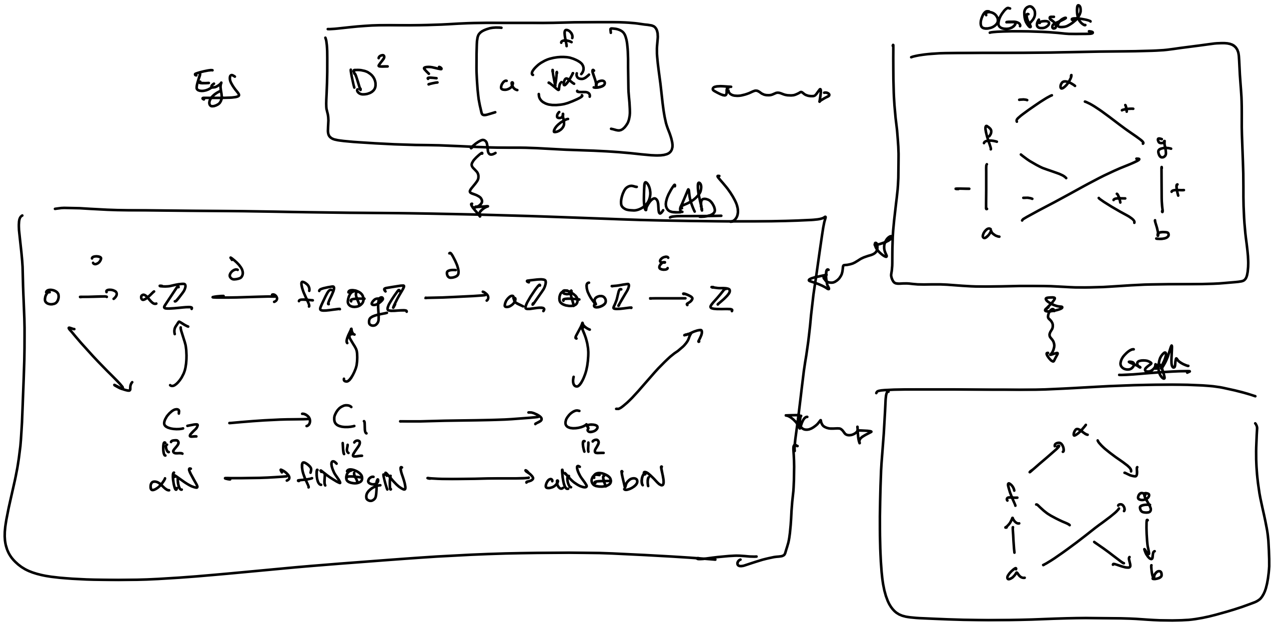
$$よ_X \otimes_{\text{Day}} よ_Y \cong よ_{X \otimes_{\mathcal{C}} Y} \qquad \square$$

3.a: Join for ∞-categories.

IDEA: there's a join construction for (Steiner) ADC's that we transport over to Steiner categories.

(Recall?) Every ADC corresponds to an "oriented Graded poset" of generating cells & boundary rel'ns.

Eg)



The cartesian product of posets can be "woven into" the equivalence:

$$\mathbf{Poset}_{\perp} \underset{(-)_{\not\perp}}{\overset{(-)_{\perp}}{\rightleftarrows}} \mathbf{Poset} \quad \cong$$

to define the "cellular join" of two posets: $X \star Y = (X_{\perp} \times Y_{\perp})_{\not\perp}$

with grading determined by:

$$\deg z = \begin{cases} \deg(x) & z = (x, \perp),\ x \in X \\ \deg(y) & z = (\perp, y),\ y \in Y. \\ \deg(x) + \deg(y) + 1 & z = (x, y),\ x \in X, y \in Y. \end{cases}$$

and orientations can also be added ("positive augmentation" − $x\ y\ z$ over $\perp$ with $+$ + some rule for $(x,y) \to (x,y')$. like $(-1)^{\dim(x) \dim(y)\dim(x,y)+1}$ or smth...)

So this definition extends to oriented graded posets, $\underline{\mathrm{OGPos}}$.

$\underline{\text{FACTS}}$: (i) The cellular join defines a monoidal structure $(\underline{\mathrm{Pos}}, \star, \emptyset)$ such that

$$(\underline{\mathrm{Pos}}_{\perp}, \times, \{\perp\}) \xrightarrow[(-)_{\not\perp}]{} (\underline{\mathrm{Pos}}, \star, \emptyset)$$

is strong monoidal.

(ii) $-\star X$, $X \star -$ preserve the initial obj. and connected colims.

(iii) The oriented 'thin' graded posets, $\underline{\mathrm{OGPos}}_t$,

all codimension 2 elements have square intervals:

$\forall x, y \in X$ s.t. $\operatorname{codim}(x,y) = 2$, $\exists!\ z_1, z_2 \in X$ s.t.

$$\begin{bmatrix} \begin{tikzcd} & y & \\ z_1 \arrow[ur] & & z_2 \arrow[ul] \\ & x \arrow[ul] \arrow[ur] & \end{tikzcd} \end{bmatrix} = [x, y] \subseteq X .$$

are the domain of the functor

$$\begin{tikzcd} \underline{\mathrm{OGPos}}_t \arrow[r, "\mathbb{Z}-"] \arrow[dr, "\mathbb{Z}-"'] & \mathrm{Ch}^+(\underline{\mathrm{Ab}}) \\ & \mathrm{DCh}^+(\underline{\mathrm{Ab}}) \arrow[u, hook, "\mathcal{U}"'] \end{tikzcd}$$

P

[AMAR'S BOOK - 11.1.6]

$$\begin{bmatrix} (\mathbb{Z}P) \\ \uparrow \\ (\mathbb{N}P) \end{bmatrix} = \begin{bmatrix} \begin{tikzcd} \cdots \arrow[r] & \mathbb{Z}P_{n+1} \arrow[r] & \mathbb{Z}P_n \arrow[r] & \cdots \arrow[r] & \mathbb{Z}P_0 \arrow[r, "\varepsilon"] & \mathbb{Z} \\ \cdots \arrow[r] & \mathbb{N}P_{n+1} \arrow[r] \arrow[u, hook] & \mathbb{N}P_n \arrow[r] \arrow[u, hook] & \cdots \arrow[r] & \mathbb{N}P_0 \arrow[u, hook] & \end{tikzcd} \end{bmatrix}$$

(iv) Join injects into Gray tensor of suspensions:

$X, Y \in \underline{\mathrm{OGPos}}$; $X \star Y \hookrightarrow S(X) \otimes_{\mathrm{Gray}} S(Y)$ [Lemma 7.4.14]

For $\mathbb{D}^1 = (a \xrightarrow{f} b)$; $\deg a = 0 = \deg b$, $\deg f = 1$.

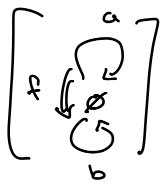
the OG poset has oriented Hasse diagram

$$H_1 = \left[\begin{tikzcd} & f \arrow[dl, dash, "-"'] \arrow[dr, dash, "+"] & \\ a & & b \end{tikzcd}\right]$$

Then

$$\underbrace{H_1 \cup \perp = \left[\begin{tikzcd} & f \arrow[dl, dash, "-"'] \arrow[dr, dash, "+"] & \\ a \arrow[dr, dash, "+"'] & & b \arrow[dl, dash, "+"] \\ & \varnothing & \end{tikzcd}\right]}_{\text{(convention)}}$$

; $\underbrace{\begin{array}{l} \deg \varnothing = 0 \\ \deg a = 1 = \deg b \\ \deg f = 2 \end{array}}$



$\cong B(\mathbb{D}^1)$

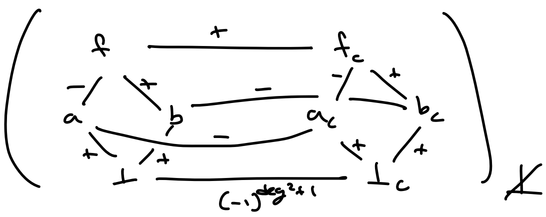
The "cellular join" of posets:

$$X \star Y = \big(\underbrace{X_\perp \times Y_\perp}_{\text{(product poset)}}\big)_{\not\perp}$$

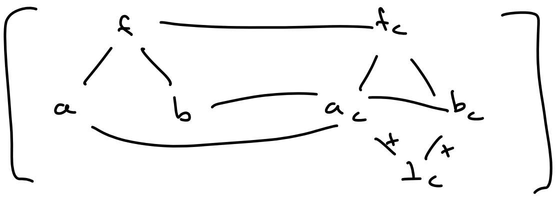
eg) $\mathbb{D}^1 \star \mathbb{D}^0 = \left(\begin{tikzcd} & f \arrow[dl, dash, "-"'] \arrow[dr, dash, "+"] & \\ a & & b \end{tikzcd}\right) \star \big(c\big)$ ← These are oriented Hasse diagrams

$$= \left[\left(\begin{tikzcd} & f \arrow[dl, dash, "-"'] \arrow[dr, dash, "+"] & \\ a \arrow[dr, dash, "+"'] & & b \arrow[dl, dash, "+"] \\ & \perp & \end{tikzcd}\right) \times \left(\begin{tikzcd} c \arrow[d, dash, "+"] \\ \perp \end{tikzcd}\right)\right]_{\not\perp}$$

$\cong$



$\cong$



$$\star \quad \left[\begin{array}{l} \left.\begin{array}{l} \dim f = 1 \\ \dim a = 0 = \dim b \end{array}\right\} \text{ as in } \mathbb{D}^1 \\ \left.\dim \perp_c = 0 \right\} \text{ as in } \mathbb{D}^0 \\ \left[\begin{array}{llll} \dim f_c & = \dim f + \dim c + 1 & = 2 & \\ \dim a_c & = \dim a + \dim c + 1 & = 1 & = \dim b_c \end{array}\right] \end{array}\right]$$

$$\text{So} \quad \left(\mathbb{D}^1 \star \mathbb{D}^0 \simeq \mathbb{D}^2\right) \simeq \left[\begin{tikzcd} a \arrow[d, "f"'] \arrow[rr, "a_c"] & \arrow[d, phantom, "\Downarrow f_c"] & c \\ b \arrow[rru, "b_c"'] & & \end{tikzcd}\right]$$

We could equivalently compute this in $\mathrm{DCh}^+(\underline{\mathrm{Ab}})$ using the "join formula".

$$\begin{tikzcd} \text{``}\underline{(X\times Y) \amalg (X\times Y)} \cong \underbrace{X \times Y \times (\partial I)}\text{''} & \text{``}\underbrace{X} \times \underbrace{Y} \times \underbrace{I}\text{''} \\ \mathbb{Z}\mathbb{D}^1 \otimes \mathbb{Z}\mathbb{D}^0 \otimes \mathbb{Z}\partial\mathbb{D}^1 \arrow[r] \arrow[d] \arrow[d, phantom, "\cong"', shift right=8] & \mathbb{Z}\mathbb{D}^1 \otimes \mathbb{Z}\mathbb{D}^0 \otimes \mathbb{Z}\mathbb{D}^1 \arrow[d] \\ \underbrace{\mathbb{Z}\mathbb{D}^1 \oplus \mathbb{Z}\mathbb{D}^0}_{\text{``}X \amalg Y\text{''}} \arrow[r] & \mathbb{Z}\mathbb{D}^1 \star \mathbb{Z}\mathbb{D}^0 \end{tikzcd}$$

$$\left(\mathbb{Z}\mathbb{D}^1 \otimes \mathbb{Z}\mathbb{D}^0\right) \oplus \left(\mathbb{Z}\mathbb{D}^1 \otimes \mathbb{Z}\mathbb{D}^0\right)$$

<u>Note:</u>

$$\mathbb{Z}\mathbb{D}^0 = \quad 0 \to \mathbb{Z}_c \xrightarrow{\epsilon} \mathbb{Z}$$

$$\mathbb{Z}\mathbb{D}^1 = \quad 0 \to \mathbb{Z}_f \to \mathbb{Z}_a \oplus \mathbb{Z}_b \xrightarrow{\epsilon} \mathbb{Z}$$

$$\mathbb{Z}\mathbb{D}^1 \otimes \mathbb{Z}\mathbb{D}^0 = \quad 0 \to \mathbb{Z}_f \to \mathbb{Z}^3_{a,b,c} \xrightarrow{\epsilon} \mathbb{Z}$$

$$\mathbb{Z}\mathbb{D}^1 \otimes \mathbb{Z}\mathbb{D}^0 \otimes \mathbb{Z}\mathbb{D}^1 = \quad 0 \to \mathbb{Z} \to \mathbb{Z}^4 \to \mathbb{Z}^4 \xrightarrow{\epsilon} \mathbb{Z}$$

$$\overbrace{(\mathbb{Z}_a\oplus\mathbb{Z}_b)\otimes_{\mathbb{Z}}(\mathbb{Z}_0\oplus\mathbb{Z}_1)}^{1}\otimes_{\mathbb{Z}}\mathbb{Z}_c \cong (\mathbb{Z}_{ac}\oplus\mathbb{Z}_{bc})\otimes_{\mathbb{Z}}(\mathbb{Z}_0\oplus\mathbb{Z}_1)$$

$$\cong \big((\mathbb{Z}_{ac}\oplus\mathbb{Z}_{bc})\otimes_{\mathbb{Z}}\mathbb{Z}_0\big)\oplus\big((\mathbb{Z}_{ac}\oplus\mathbb{Z}_{bc})\otimes_{\mathbb{Z}}\mathbb{Z}_1\big)$$

$$\cong (\mathbb{Z}_{ac0}\oplus\mathbb{Z}_{bc0})\oplus(\mathbb{Z}_{ac1}\oplus\mathbb{Z}_{bc1}) \cong \big(\underline{\mathbb{Z}_{ac0}\oplus\mathbb{Z}_{bc0}\oplus\mathbb{Z}_{ac1}\oplus\mathbb{Z}_{bc1}}\big)$$

$$\cong \mathbb{Z}^4$$

$$\overbrace{\big(\mathbb{Z}_f\otimes_{\mathbb{Z}}\mathbb{Z}_c\otimes_{\mathbb{Z}}(\mathbb{Z}_a\oplus\mathbb{Z}_b)\big)\oplus\cancel{\big((\mathbb{Z}\oplus\mathbb{Z})\otimes_{\mathbb{Z}}0\otimes_{\mathbb{Z}}(\mathbb{Z}\oplus\mathbb{Z})\big)}\oplus\big((\mathbb{Z}_a\oplus\mathbb{Z}_b)\otimes_{\mathbb{Z}}\mathbb{Z}_c\otimes_{\mathbb{Z}}\mathbb{Z}_f\big)}$$

$$\cong (\mathbb{Z}_{fca}\oplus\mathbb{Z}_{fcb})\oplus 0\oplus(\mathbb{Z}_{acf}\oplus\mathbb{Z}_{bcf}) \cong \big(\underline{\mathbb{Z}_{fca}\oplus\mathbb{Z}_{fcb}\oplus\mathbb{Z}_{acf}\oplus\mathbb{Z}_{bcf}}\big)$$

$$\cong \mathbb{Z}^4$$

$$(\mathbb{Z}D^1\otimes_{\mathbb{Z}}\mathbb{Z}D^0)\cong 0\longrightarrow\mathbb{Z}_{fc}\longrightarrow\mathbb{Z}_{ac}\oplus\mathbb{Z}_{bc}\xrightarrow{\epsilon}\mathbb{Z}$$

$$(\mathbb{Z}D^1\otimes_{\mathbb{Z}}\mathbb{Z}D^0)\oplus(\mathbb{Z}D^1\otimes_{\mathbb{Z}}\mathbb{Z}D^0)\cong 0\longrightarrow\underset{\cancel{\mathbb{Z}_{fc}\oplus\mathbb{Z}_{(fc)'}}}{\mathbb{Z}^2}\longrightarrow\underset{\cdots}{\mathbb{Z}^4}\xrightarrow{\epsilon}\mathbb{Z}$$

So our pushout (in $Ch^f(Ab)$), is given by:

$$\begin{tikzcd}
{[0\to\underset{fc}{\mathbb{Z}^2}\to\underset{\substack{ac\\bc}}{\mathbb{Z}^4}\xrightarrow{\epsilon}\mathbb{Z}]} \arrow[r] \arrow[d, "\substack{(fc)_1\\(ac)1\\(bc)1}"'] & {[0\to\underset{fcf}{\mathbb{Z}}\to\underset{\substack{fca\\fcb\\acf\\bcf}}{\mathbb{Z}^4}\to\underset{\substack{ac0\\bc0\\ac1\\bc1}}{\mathbb{Z}^4}\xrightarrow{\epsilon}\mathbb{Z}]} \arrow[d] \\
{[0\to\underset{f}{\mathbb{Z}}\to\underset{\substack{a\\b\\c}}{\mathbb{Z}^3}\xrightarrow{\epsilon}\mathbb{Z}]} \arrow[r] & {[0\to\mathbb{Z}^{1+0-0}\to\mathbb{Z}^{4+1-2}\to\mathbb{Z}^{4+3-4}\xrightarrow{\epsilon}\mathbb{Z}]} \arrow[d, phantom, "\wr\!\!\parallel"] \\
 & {[0\to\underset{fc}{\mathbb{Z}}\to\underset{\substack{f\\ac\\bc}}{\mathbb{Z}^3}\to\underset{\substack{a\\b\\c}}{\mathbb{Z}^3}\xrightarrow{\epsilon}\mathbb{Z}]}
\end{tikzcd}$$

If we write out the generators too, we can see

$$
\begin{array}{ccc}
\left[\begin{tikzcd} ac \arrow[d, "fc"'] & & ac_1 \arrow[d, "fc_1"] \\ bc & & bc_1 \end{tikzcd}\right] & \longrightarrow & \left[\begin{tikzcd} ac0 \arrow[r, "acf"] \arrow[d, "fca"'] \arrow[dr, phantom, "\Swarrow fcf"] & ac1 \arrow[d, "fcb"] \\ bc0 \arrow[r, "bcf"'] & bc1 \end{tikzcd}\right] \\
\downarrow & & \downarrow \\
\left[\begin{tikzcd} a \arrow[d, "f"'] & & c \\ b & & \end{tikzcd}\right] & \longrightarrow & \left[\begin{tikzcd} a \arrow[d, "f"'] \arrow[dr, "a_c"] \arrow[dr, phantom, "\Downarrow f_c", shift right=3ex] & \\ b \arrow[r, "b_c"'] & c \end{tikzcd}\right]
\end{array}
$$

The relations are:

$$
\left\{\begin{array}{l} ac_0 = a \\ bc_0 = b \\ fca = f \end{array}\right.
$$

$$
\left\{ ac_1 = c = bc_1 \,,\ fcb = 0 \right.
$$

$$
\left\{\begin{array}{l} acf = a_c \\ bcf = b_c \\ fcf = f_c \end{array}\right.
$$

__THM__: For any full subcat $\mathcal{C} \hookrightarrow \underline{\infty Cat}^{Steiner}$ which inherits the $\circledast$-monoidal structure and is dense in $\underline{\infty Cat}$

The Day-convolution monoidal structure on $P_{red}(\mathcal{C})$ descends to the localization

$$\begin{tikzcd} ① \rightsquigarrow & P_{red}(\mathcal{C}) \arrow[r, shift left, "T"] \arrow[r, phantom, "\perp"] & \underline{\infty Cat} \arrow[l, shift left, "N"'] \\ ② \rightsquigarrow & P_{red}(\mathcal{C}) \otimes \underline{Set} \arrow[u] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \underline{\infty Cat}^{Strict} \arrow[u, shift left] \arrow[l, shift left] \arrow[u, shift right, leftarrow] \end{tikzcd}$$

Before we prove this, we need to know a bit about $P_{red}(\mathcal{C})$ ...

Assume $\mathcal{C}$ has an initial object, $\varnothing$.

__Def__: A pshf $X : \mathcal{C}^{op} \longrightarrow \mathcal{S}$ is __REDUCED__ if $X(\varnothing) = *$ is terminal.

__Fact__: $P_{red}(\mathcal{C}) \underset{\perp}{\overset{L}{\leftrightarrows}} P(\mathcal{C})$ can be described as the full subcategory of local objects wrt the singleton

$$\{ \varnothing \hookrightarrow \mathcal{X}_\varnothing \}$$

$$\varnothing(c) = \varnothing \;\; \forall c \in \mathcal{C}.$$

i.e.

$$\begin{tikzcd} \varnothing \arrow[r, "\text{“}\forall\text{”}"] \arrow[d, hook] & X \\ \mathcal{X}_\varnothing \arrow[ur, dashed, "\exists ! \theta"'] & \end{tikzcd}$$

The is essentially only one of these.

$$\theta_\varnothing : \mathcal{X}_\varnothing(\varnothing) \longrightarrow X(\varnothing)$$
$$\mathcal{X}_\varnothing(\varnothing) \cong (*) \dashrightarrow X(\varnothing)$$

but $*$ is terminal

$X(\varnothing) \simeq *$

Examples: (0) "$\Delta_+^{op} \xrightarrow{X} Set$ is reduced" $\iff$ "X is the trivial augmentation of its underlying simp'l set."

(i) $\mathcal{L}_\emptyset = \mathcal{C}(-,\emptyset)$ is reduced

$$\mathcal{C}(X,\emptyset) = \begin{cases} \emptyset & \text{if } X \cong \emptyset \\ X & \text{else} \end{cases}$$

(ii) Any rep'ble $\mathcal{L}_Y = \mathcal{C}(-,Y)$ is reduced

$$\mathcal{C}(\emptyset, Y) \cong * \quad \text{for all } Y.$$

i.e.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathcal{L}"] \arrow[dr, dashed] & \mathcal{P}(\mathcal{C}) \\ & \mathcal{P}_{red}(\mathcal{C}) \arrow[u] \end{tikzcd}$$

(iii) Small weakly contractible colimits of reduced presheaves are reduced ...

(small, weakly contractible) $\rightsquigarrow$ $I \xrightarrow{X} \mathcal{P}_{red}(\mathcal{C})$

$$\left[(i \xrightarrow{f} j) \right]_I \longrightarrow \left[\begin{tikzcd} X_i \arrow[r, "X_f"] \arrow[dr] & X_j \arrow[d] \\ & \operatorname{colim} X_\alpha \end{tikzcd} \right]_{\mathcal{P}_{red}(\mathcal{C})}$$

If you "evaluate" the diagram at $\emptyset$, it contracts to $*$ ...

$$\left[\begin{tikzcd} * \arrow[r, phantom, "\simeq"] \arrow[rrrr, bend left, "\overset{id.}{\simeq}"] \arrow[drr, bend right, "\simeq"'] & X_i(\emptyset) \arrow[rr, "X_f(\emptyset)"] \arrow[dr] & & X_j(\emptyset) \arrow[dl] \arrow[r, phantom, "\simeq"] & * \arrow[dll, bend left, "\simeq"] \\ & & \operatorname{colim} X_\alpha \arrow[u, phantom, "\simeq"] & & \end{tikzcd} \right]_{\mathcal{S}}$$

colim X_α is built by gluing the diagram X into a single object ... If the diagram is weakly contractible, then the $\emptyset$-cells of the colimit are built by gluing the $\emptyset$-cells of the diagram together ... but the diagram <u>contracts</u> at $\emptyset$!

Kerodon

$\boxed{\text{Cor } 4.7.23}$: If $\mathcal{C}$ has an initial object, then $\mathcal{C}$ is weakly contractible.

SKETCH

① $\perp \in \mathcal{C}$ is initial iff

$$\begin{tikzcd} \mathcal{C}_{\perp/} \arrow[r, "p"] & \mathcal{C} \arrow[l, dashed, bend left, "\sigma"] \end{tikzcd} \quad \text{s.t.} \quad \sigma(\perp) = id_\perp .$$

② Since $\{\perp\} \hookrightarrow \mathcal{C}$ is monic, $\{id_\perp\} \hookrightarrow \mathcal{C}_{\perp/}$ is right anodyne (has LLP wrt $\underbrace{\text{right Kan fibrations}}_{\text{eg: Grothendieck/Cartesian fibs like slices!!}}$

③ Then we get a diagram in the arrow category:

$$\begin{tikzcd} \{\perp\} \arrow[r, "!"'] \arrow[d] \arrow[rr, bend left, "id"] & \{id_\perp\} \arrow[r, "!"'] \arrow[d] & \{\perp\} \arrow[d] \\ \mathcal{C} \arrow[r, "\sigma"] \arrow[rr, bend right, "id"'] & \mathcal{C}_{\perp/} \arrow[r, "p"] & \mathcal{C} \end{tikzcd}$$

and since right anodyne maps are stable under retracts, $\{\perp\} \hookrightarrow \mathcal{C}$ is also right anodyne. In particular it's anodyne and hence a weak homotopy equiv. //

$\mathcal{C} \in \infty\mathrm{Cat}$

Generally (Daniel's sketch, if my brain still works...)

$\emptyset \in \mathcal{C}$ initial

$$\begin{tikzcd} & * \arrow[dr, "\emptyset"] \arrow[d, phantom, "\Downarrow"] & \\ \mathcal{C} \arrow[ur, "!"] \arrow[rr, equal] & {} & \mathcal{C} \end{tikzcd}$$

Then
$$\begin{tikzcd} & * \arrow[dr, bend left, "|\emptyset|"] \arrow[d, phantom, "\Downarrow"] & \\ {|N(\mathcal{C})|} \arrow[ur, "!"] \arrow[rr] & {} & {|N(\mathcal{C})|} \end{tikzcd}$$
$\in \mathcal{S}$ but 2-cells between gpds are invertible so this is an iso //

(Helper)

$\boxed{\text{Lemma}}$: The full subcat of objects $X \in \underline{\infty Cat}^{\text{Steiner}}$ such that $- \star X$, $X \star -$ preserve local equivalences

(i) contains $\emptyset$

(ii) is closed under small weakly contractible colims

(iii) is closed under all small colimits

Proof: (i) $\&_{\emptyset}$ is the monoidal unit for $\star$

(ii) these functors preserve colimits from $P_{ed}(\mathcal{C})$

(iii) Every small colimit is a (disjoint) coproduct of small weakly contractible ones ?

[3.9.4-GH]

$\boxed{\text{Lemma}}$ $\mathcal{C} \hookrightarrow \underline{\infty Cat}^{\text{Steiner}}$ full monoidal subcat, w.r.t. $\star$.

Then the Day convo product on $P(\mathcal{C})$ restricts to $P_{ed}(\mathcal{C})$

Proof: Want to see $(X \underset{\text{Day}}{\star} Y)(\emptyset) \simeq *$

* $\&: \mathcal{C} \hookrightarrow P(\mathcal{C})$ is monoidal (by coend calc)
* $\&_{\emptyset}$ is the unit for $\underset{\text{Day}}{\star}$ in $P(\mathcal{C})$. and it's in $P_{ed}(\mathcal{C})$.

$$X \star Y\ (\emptyset) \cong \int^{A,B} \mathcal{C}(\emptyset, A \star B) \times X_A \times Y_B$$

$$\cong \int^{A,B} \{*\} \times X(A) \times Y(B)$$

$$\cong \operatorname*{colim}_{A,B \in \mathcal{C}} \big(X(A) \times Y(B)\big)$$

~~$Tw(\mathcal{C})$~~? Nvm!

[GH]

Lol, of course.

$\star$ the profunctor $(\mathcal{C}\times\mathcal{C})^{op} \times (\mathcal{C}\times\mathcal{C}) \longrightarrow S$

is mute in the second variable, so the coend collapses to a colimit in the other variable

$$\frac{A, B \in \mathcal{C}}{(A,B) \in (\mathcal{C} \times \mathcal{C})^{op}}$$

$$\cong \left(\operatorname{colim}_A X(A)\right) \times \left(\operatorname{colim}_B X(B)\right)$$

$$\cong X(\emptyset) \times Y(\emptyset)$$

$$\cong *$$

Proof of Thm 3.9 - ①

We want to show that the restricted join-monoidal structure on $\underbrace{\mathcal{C} \hookrightarrow \underline{\infty Cat}^{Steiner}}_{full}$ descends to the localization

$$P_{red}(\mathcal{C}) \underset{L}{\overset{\top}{\rightleftarrows}} \underline{\infty Cat}$$

if $\mathcal{C}$ is dense in $\underline{\infty Cat}$.

Base Case: $\left(\mathcal{C} = \underline{\infty Cat}^{Steiner}\right)$

We need to see that the functors

$$- \star X, \quad X \star -$$

preserve local equivalences, for any $X \in P_{red}(\mathcal{C})$.

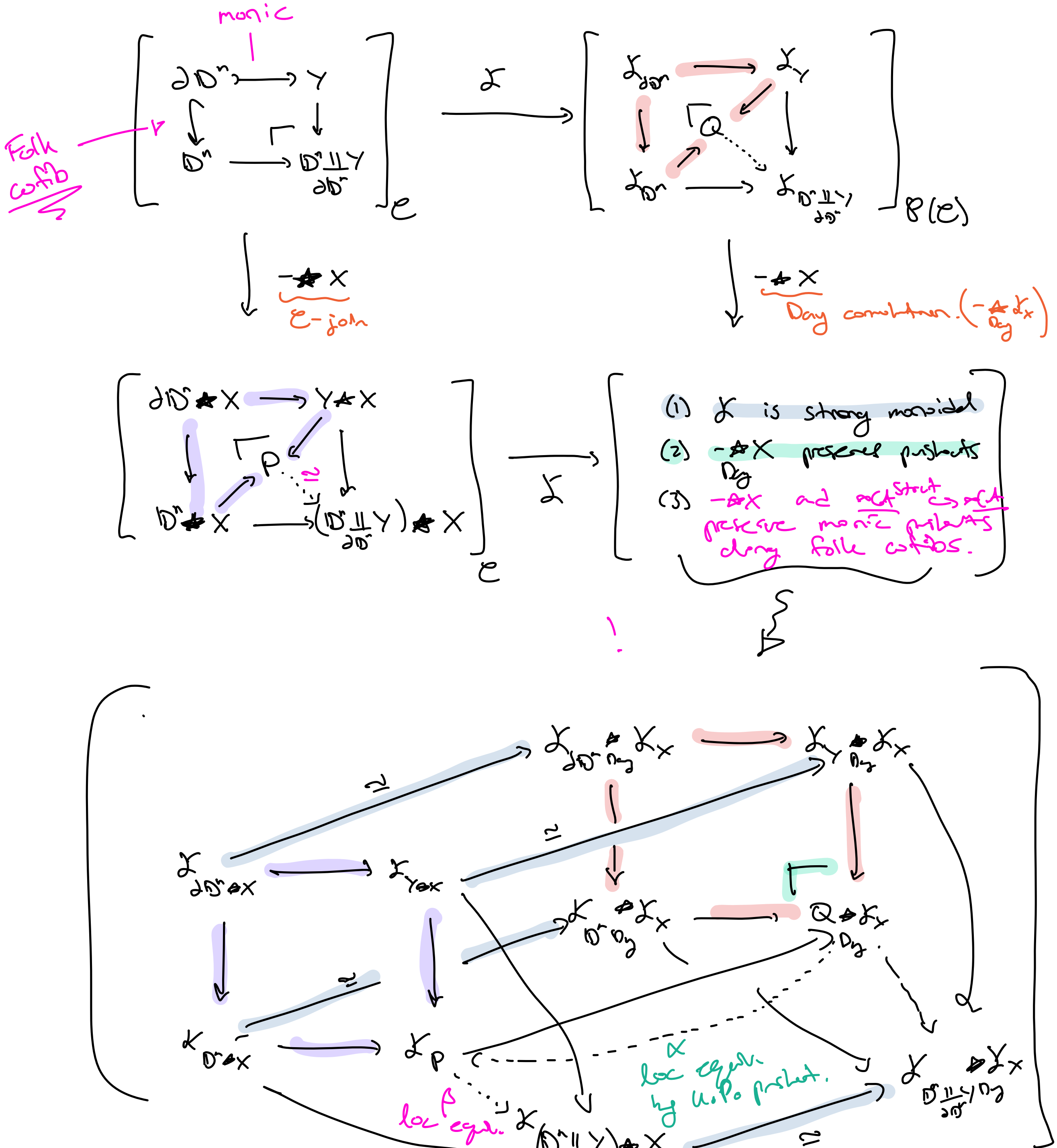
By Helper lemma, the full subcat of all such objects X is closed under small colimits.

By Density $P_{red}(\underline{\infty Cat}^{Steiner}) = \langle \text{よ}_X \mid \text{small colims} \rangle$

so reduce to the case when

* X is rep'ble (Abuse: $X = \text{よ}_X$)
* $Y = \mathbb{D}^n \coprod_{\partial \mathbb{D}^n} Z$ for some $Z \in \underline{\infty Cat}^{Steiner}$. ("$Y = \text{よ}_Y$" here).

and $\mathbb{D}^n \hookrightarrow Y$ (then $L\mathbb{D}^n \xrightarrow{\cong} LY$?)



This shows base case

Now let
$$\begin{tikzcd} \mathcal{C} \arrow[r, hook, "i"] \arrow[rr, bend left, hook, "\text{(dense)}"] & \underline{\infty Cat}^{Steiner} \arrow[r, hook, "j"] & \underline{\infty Cat} \end{tikzcd}$$

(monoidal, full) (dense)

$\mathcal{C}$ is dense in ∞Cat, so the restricted nerve $N|_{\mathcal{C}}$ is f.f. this means we can factor $N|_{Steiner}$ through $N|_{\mathcal{C}}$ via right Kan extension

$$\begin{tikzcd} \underline{\infty Cat} \arrow[r, "y"] \arrow[rr, bend left, "N|_{Steiner}", "\underline{def}"'] \arrow[dr, "N|_{\mathcal{C}}"'] & P(\underline{\infty Cat}) \arrow[r, "j^*"] & P(\underline{\infty Cat}^{Steiner}) \arrow[dl, bend right, "i^*"'] \\ & P(\mathcal{C}) \arrow[ur, bend right, "ran_i"'] \arrow[ur, phantom, "\perp"] & \end{tikzcd}$$

ran$_i$ then

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[r, "X"] \arrow[d, "i"'] & \mathcal{S} \\ \underline{\infty Cat}^{Steiner\, op} \arrow[ur, dashed, "ran_i(X)"'] \arrow[ur, phantom, "\Uparrow"] & \end{tikzcd}$$

* By adjointness $i^* \dashv ran$, $L_{Steiner}$ factors through $L_{\mathcal{C}}$

Adjoints compose & are unique (up to equiv)

$$\begin{tikzcd} \underline{\infty Cat} \arrow[r, "y"] \arrow[rr, bend left, "\underline{def}"] \arrow[dr, bend right, "N|_{\mathcal{C}}"'] & P(\underline{\infty Cat}) \arrow[r, "j^*"] & P(\underline{\infty Cat}^{Steiner}) \arrow[dl, bend right, "i^*"'] \arrow[ll, bend right=60, "L_{Steiner}"', "\perp"] \\ & P(\mathcal{C}) \arrow[ul, "L_{\mathcal{C}}", "\perp"'] \arrow[ur, bend right, "ran_i"'] \arrow[ur, phantom, "\perp"] & \end{tikzcd}$$

and i^* restricts to P_{red}. so similar for reduced :)

$$\begin{tikzcd} P_{red}(\underline{\infty Cat}^{Steiner}) \arrow[r, hook] \arrow[d, "i^*"'] \arrow[rr, bend left, "L_{Steiner_{red}}"] & P(\underline{\infty Cat}^{Steiner}) \arrow[r, "L_{Steiner}"] \arrow[d, "i^*"'] & \underline{\infty Cat} \\ P_{red}(\mathcal{C}) \arrow[r] \arrow[rr, bend right, "L_{\mathcal{C}_{red}}"'] & P(\mathcal{C}) \arrow[ur, "L_{\mathcal{C}}"'] & \end{tikzcd}$$

Thus $L_{\mathrm{Steiner}} \circ i_!$ factors as $L_{\mathcal{C}_{\mathrm{red}}} \circ i^* \circ i_! = L_{\mathcal{C}_{\mathrm{red}}}$.

$$\begin{tikzcd}
\mathcal{P}_{\mathrm{red}}(\infty\mathrm{Cat}^{\mathrm{Steiner}}) \arrow[r, hook] \arrow[d, "i^*"] \arrow[rr, bend left, "L_{\mathrm{Steiner}_{\mathrm{red}}}"] & \mathcal{P}(\infty\mathrm{Cat}^{\mathrm{Steiner}}) \arrow[r, "L_{\mathrm{Steiner}}"] \arrow[d, "i^*"] & \underline{\infty\mathrm{Cat}} \\
\underbrace{\mathcal{P}_{\mathrm{red}}(\mathcal{C})}_{} \arrow[u, bend left, "i_!"] \arrow[r] \arrow[rr, bend right, "L_{\mathcal{C}_{\mathrm{red}}}"'] & \mathcal{P}(\mathcal{C}) \arrow[ur, "L_{\mathcal{C}}"'] &
\end{tikzcd}$$

$\mathcal{C} \xhookrightarrow{i} \underline{\infty\mathrm{Cat}}^{\mathrm{Steiner}}$ is f.f.

$$\begin{tikzcd}
\mathcal{P}(\mathcal{C}) \arrow[r, bend left, "i_!"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, bend right, "i_*"'] \arrow[r, phantom, shift right=2, "\perp"] & \mathcal{P}(\infty\mathrm{Cat}^{\mathrm{Steiner}}) \arrow[l, "i^*"']
\end{tikzcd}$$

and $i_!$ and i_* are also f.f. [Lemma 3.2 nLab – functoriality of cat's of pshvs]

Unit $\mathbb{1} \overset{\eta}{\Longrightarrow} i^* \circ i_!$ of $i_! \dashv i^*$ (and counit of $i^* \dashv i_*$) are isomorphisms.

<u>Finally</u>:

$$\underbrace{\mathcal{P}_{\mathrm{red}}(\mathcal{C}) \xrightarrow{i_!} \mathcal{P}_{\mathrm{red}}(\infty\mathrm{Cat}^{\mathrm{Steiner}})}_{} \underbrace{\xrightarrow{L} \underline{\infty\mathrm{Cat}}}_{}$$

[$\otimes$-monoidal by Lem. 3.9.4] [$\otimes$-monoidal by these are]

$\boxed{\star}$

Application(s)

[Cor 3.9.8 GHT] $\longrightarrow$ Take $\mathcal{C} = \Delta^+$ full subcat spanned by orientals & $\varnothing$.

$$\begin{tikzcd} \underline{\infty Cat}^{\text{Steiner}} \arrow[r, hook] \arrow[d, "- \star X"'] & \underline{\infty Cat} \arrow[d, "- \star X"] \\ \underline{\infty Cat}^{\text{Steiner}} \arrow[r, hook] & \underline{\infty Cat} \end{tikzcd}$$

are $\star$-monoidal & compatible with weakly compatible colimits and $\varnothing \star X \simeq X$.

(Tensor unit)

This becomes monoidal if we instead map to $\underline{\infty Cat}_{X/}$

$$\begin{aligned} \underline{\infty Cat} &\longrightarrow \infty Cat_{X/} \\ A &\longmapsto (X \xhookrightarrow{i} A \star X) \\ \varnothing &\longmapsto (X = X) \end{aligned}$$

and presentability of $\underline{\infty Cat}$ gives a right adjoint

$$\begin{tikzcd} Cat_{X/} \arrow[r, shift right, "X /\!/^{oplax}_{(-)}"'] \arrow[r, phantom, "\perp"] & \underline{\infty Cat} \arrow[l, shift right, "- \star X"'] \end{tikzcd}$$

$(F : X \to Y) \longmapsto X /\!/^{oplax}_{F}$ the oplax slice/over category

(+ variations & comparisons of 'op' & 'co')...

Eg: $\mathrm{Fun}\Big(\underbrace{\mathbb{D}^0 \star \mathbb{D}^1, X}_{\text{(oplax) cones in } X}\Big) \simeq \mathrm{Fun}\Big(\underbrace{\mathbb{D}^1, \mathbb{D}^0 /\!/^{oplax}_{X}}_{\text{Maps in oplax slice}}\Big)$

3.10 Join Formula:

For any ∞-cats X, Y, there's a pushout (in $\underline{\infty Cat}$)

$$\begin{tikzcd} X \boxtimes Y \amalg X \boxtimes Y \arrow[r] \arrow[d] & X \boxtimes D^1 \boxtimes Y \arrow[d] \\ X \amalg Y \arrow[r] & X \diamond Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

THM: $\underbrace{X \diamond Y}_{\text{(pushout)}} \simeq \underline{X \star Y}$ (my conv of Steiner join descended to $\underline{\infty Cat}$.)

Pf: For X, Y Steiner cats, this exists in $\underline{\infty Cat}^{\text{strict}}$ (where the join is the ext'n of Steiner join to $\underline{\infty Cat}$)

This join def'n, and the pushout def'n above ($X \diamond Y$), preserve $\frac{\emptyset,\ \text{w.c. colims}}{\text{all colims}}$ } in each var. sep.

By density of Steiner cats & naturality of $X \diamond Y$ the po. diagram extends to $\underline{\infty Cat}$ for all X, Y.

(Do $- \star Y$ first, then $X \star Y$; one var. at a time extn...)

This induces a functor

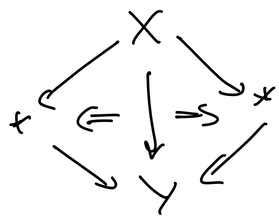
$$X \diamond Y \xrightarrow{\rho} X \star Y \qquad \text{nat'l in } X, Y.$$

To see ρ is an equiv. it suffices to check case $X, Y \in \Delta$ (Δ is dense, the functors above preserve colimits)

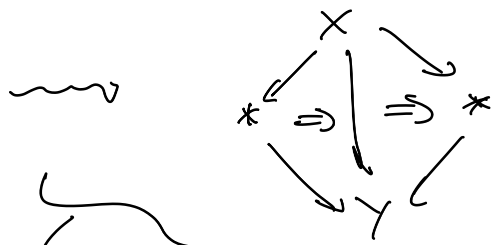
$\vdots$

(Some pushout lemmas ...) Ref [GH]

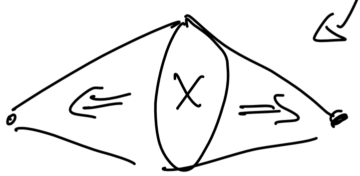
Bicones

"  " ?

Orthoplexes



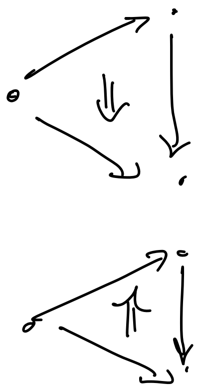
These look like functors
I define in my thesis...

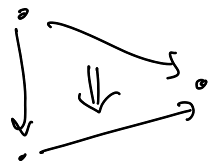


and in a cat. of functors
we imagine doing this
with <u>some</u> "<u>left legs</u>".

So <u>maybe</u> ... these aren't walking cells in
our cats as much as functors of those?

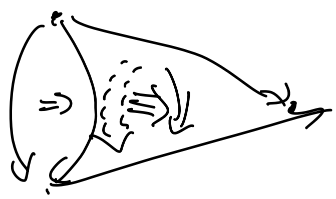
↳ I am actually working on/around this
in my thesis...

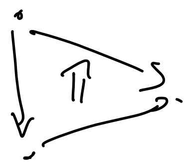




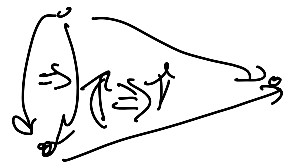
$(\mathbb{D}^1 \star \mathbb{D}^0)$

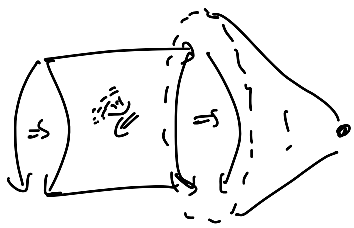
$\mathbb{D}^R \star \mathbb{D}^0$

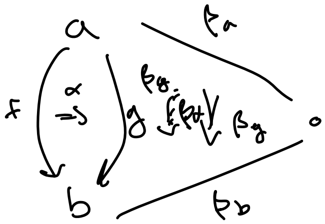




$(\mathbb{D}^1 \star \mathbb{D}^0)^{co}$

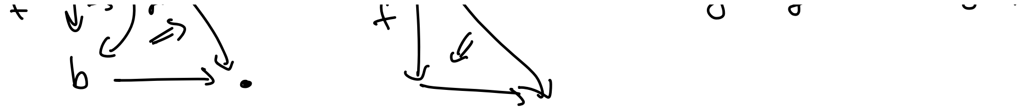






src(xy) = src(x) y

tgt (xy) = x tgt y ?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